Math 615, Winter 2011 **Problem Set #4**
Due: Monday, April 11

1. Let x be a nonzerodivisor on a Noetherian module M over a Noetherian ring. Suppose that $\dim(M) = n$. Show that $xH^n_I(M) = H^n_I(M)$. Hence, if R is a local domain of dimension n, $H^n_I(R)$ is a divisible R-module.

2. Let $S = K[[x_1, \ldots, x_n]]$ denote formal power series in n variables over a field K, where $1 \leq k \leq n$ are integers Let R be the subring of S consisting of power series in which the total degree of every monomial that occurs is divisible by k. Let $A = k[[x_1^k, \ldots, x_n^k]] \subseteq R$.

(a) Explain why $\mathrm{Hom}_A(S, A) \cong S$ as an S-module, and give an explicit generator.

(b) What is the type of R?

3. Let R be a Cohen-Macaulay local domain and I a proper ideal of R such that $I \cong \omega_R$ as R-modules. Prove that R/I is a Gorenstein local ring.

4. Let K be a field and let $R = K[x, y, u, v]$ denote a polynomial ring in 4 variables over R, let $m = (x, y, u, v)R$.Let $f = xu$, $g = yv$, and $h = xv + yu$. Let $I = (f, g, h)$.

(a) Determine the minimal primes of I, the radical J of I, and show that I has height 2.

(b) Show that $H^3_I(R) \cong H^4_m(R) \cong E_R(R/m) \neq 0$. [Use the Mayer-Vietoris theorem.]

5. Let $S = K[[x_1, \ldots, x_n]]$ be a formal power series, ring, and let $I = (x_i x_j : i \neq j)$. Let $R = S/I$. $\dim(R) = 1$ and R is reduced $\Rightarrow$ Cohen-Macaulay. What is the type of R?

6. Let $X = (x_{ij})$ denote a 3×2 matrix over a field K. Let $S = K[x_{ij} : i, j]$ be the polynomial ring in the 6 variables x_{ij} over K. Let Δ_i denote $(-1)^{i-1}$ times the 2×2 minor of X obtained by omitting the ith row. Let $I = I_2(X)$ denote the ideal of S generated by the Δ_i, and let $R = S/I$. You may assume thar R is a Cohen-Macaulay domain of dimension 4. Let $M = \big(\Delta_1\ \Delta_2\ \Delta_3\big)$. You may assume that $0 \longrightarrow S^2 \xrightarrow{X} S^3 \xrightarrow{M} S \longrightarrow R \longrightarrow 0$ is exact. Prove that $\omega = (x_{11}, x_{12})R$ is a global canonical module for R.

EC5. Let $r \leq s$ be integers, let $X = (x_{ij})$ be an $r \times s$ matrix of indeterminates, and let $S = K[[x_{ij} : i, j]]$ be the formal power series ring in the rs variables x_{ij} over K. Let $P = I_r(X)$ denote the ideal generated by the $r \times r$ minors of X. Let $R = S/P$. You may assume that this ring is Cohen-Macaulay of dimension $(r-1)s + r - 1 = rs - (s - r + 1)$. (It is also known to be a domain.)

(a) The matrix X has $s - r + 1$ diagonals D_i, $1 \leq i \leq s - r + 1$, where D_i begins with the element $x_{1,i}$ of the first row and contain the elements $x_{1+k,i+k}$ for $0 \leq k \leq r - 1$. Show that the elements below the first diagonal (x_{ij} for $i > j$), above the last diagonal (x_{ij} for $j > s - r + i$), together with the differnces of the consecutive elements on these diagonals ($x_{1+k+1,i+k+1} - x_{1+k,i+k}$, $1 \leq i \leq s - r + 1$, $0 \leq k \leq r - 2$) form a system of parameters for R.

(b) Show that the quotient of R by the ideal generated by the system of parameters described above is isomorphic to $T/\mathcal{M}^r$, where T is the polynomial ring in $s - r + 1$ variables and $\mathcal{M}$ is the maximal ideal of T generated by the variables.

(c) Give the type of R as a binomial coefficient.