Math 615, Winter 2011 **Problem Set #4: Solutions**

1. Since $\dim(M/xM) < \dim(M) = n$, we know from a class theorem that $H_I^n(M/xM) = 0$. Consequently, the three terms of degree n in the long exact sequence for local cohomology coming from the short exact sequence $0 \to M \xrightarrow{x} M \to M/xM \to 0$ are $H_I^n(M) \xrightarrow{x} H_I^n(M) \to 0 = H_I^n(M/xM)$, which shows the surjectivity of multiplcation by x on $H_I^n(M)$, as required. The statement about domains R is immediate. $\square$

2. (a) Since S is module-finite over A, which is regular, by a class theorem $\mathrm{Hom}_A(S,\, A) \cong \omega_S$, and this is S since S is regular. S is free over A on the basis consisting of monomials $\mu = x_1^{a_1} \cdots x_n^{a_n}$ such that $0 \le a_i \le k-1$, $1 \le i \le n$. Thus, $\mathrm{Hom}_A(S,\, A)$ has a free basis the maps T_μ, where $T_m u$ has value 1 on μ and is 0 on the other monomials in the free basis. Let $\nu = x_1^{k-1} \cdots x_n^{k-1}$. Then T_ν generates $\mathrm{Hom}_A(S,\, A)$ over S: if μ is any monomial in the free basis for S over A we can write $\nu = \rho\mu$ for some monomial ρ in S, and the $T_\mu = \rho T_\nu$, for the value of ρT_ν on a monomial λ in the free basis is $T(\rho\lambda)$. If $\lambda = \mu$ this is 1. For any other λ, $\rho\lambda$ is a monomial in A times a monomial in the free basis different from ν, and the value is 0.

(b) Since $x_1^k, \ldots, x_n^k$ is a system of parameters, it suffices to find the dimension of the socle modulo the ideal they generate. Because of the $\mathbb{N}^n$-grading, the socle will be spanned by monomials, and we need only count the number of monomials $\mu = x_1^{a_1} \cdots x_n^{a_n}$ satisfying these conditions: (1) $0 \le a_i \le k-1$, $1 \le i \le n$, (2) $\sum_i a_i \equiv 0 \bmod k$, and (3) if α is any monomial of degree k, at least one exponent of $\alpha\mu$ is at least k. Let $b_i = k - 1 - a_i$. We count the choices for the b_i instead. Condition (1) is unaffected. Condition (3) is easily seen to be equivalent to the condition that $\sum_i b_i \le k-1$ (e.g., if it is k or more, we may multiply by a degree k divisor of $x_1^{b_1} \cdots x_n^{b_n}$ without getting an exponent larger than $k-1$). Condition (2) becomes $\sum_i (k-1-b_i) \equiv 0 \bmod k$ or $\sum_i b_i \equiv -n \bmod k$. Let h denote the unique integer between 0 and $k-1$ inclusive that is congruent to $-n \bmod k$. Now condition (3) implies that $\sum_i b_i = h$. Therefore, the type is the same as the number of monomials of degree h in n variables, which is $\binom{h+n-1}{h}$. $\square$

3. Let $(R,\, m,\, K)$ have dimension n. By a class theorem, I is an ideal all of whose associated primes have height one. It follows that $\dim(R/I) = n-1$, To show that R/I is Cohen-Macaulay, it suffices to show that the depth of R/I is at least $n-1$, since it cannot exceed $\dim(R/I)$. From the short exact sequence $(*)\quad 0 \to I \to R \to R/I \to 0$ we have a long exact sequence for local cohomology part of which is $H_m^i(I) \to H_m^i(R/I) \to H_m^{i+1}(R)$. For $i < n-1$, the first and third terms vanish, since $I \cong \omega_R$ and R both have depth n. It follows that $H_m^i(R/I) = 0$ for $i < n-1$, as required. By a class theorem, if M is finitely generated Cohen-Macaulay of dimension d, then $\mathrm{Ext}^i(K,\, M) = 0$ for $i < d$ while the K-vector space dimension of $\mathrm{Ext}_R^d(K,\, M) = 0$ is the type of R. Part of the long exact obtained by applying $\mathrm{Hom}_R(K,\, \_)$ to $(*)$ is $0 = \mathrm{Ext}_R^{n-1}(K,\, R) \to \mathrm{Ext}_R^{n-1}(K,\, R/I) \to \mathrm{Ext}_R^n(K,\, I) \cong K$, since $I \cong \omega_R$ has type 1. Since $\mathrm{Ext}_R^{n-1}(K,\, R/I)$ injects into K, the type of R/I must be one. $\square$
Alternatively, by local duality, $\omega_{R/I} \cong \mathrm{Ext}_R^1(R/I,\, I)$, and applying $\mathrm{Hom}_R(\_,\, I)$ to $(*)$ yields $\cdots \to \mathrm{Hom}_R(R,\, I) \to \mathrm{Hom}_R(I,\, I) \to \mathrm{Ext}_R^1(R/I,\, R) \to \mathrm{Ext}_R^1(R,\, I) = 0$. The image of $\mathrm{Hom}_R(R,\, I) \cong I$ in $\mathrm{Hom}_R(I,\, I) \cong R$ is I, which shows that $\omega_{R/I} \cong R/I$. $\square$

4. Let $P = (x, y)$ and $Q = (u, v)$. Note that $P \cap Q = (x,\, y) \cap (u, v) = (xu,\, xv,\, yu,\, yv)$ is a radical ideal containing I, and that $(xv)^2 = xvh - fg \in I$ and $(yu)^2 = yuh - fg \in I$. Hence, $P \cap Q = J = \mathrm{Rad}\,(I)$, P and Q are the minimal primes of I, and since P and Q have height 2, so does I. Note that $P + Q = m$, the homogeneous maximal ideal of R. The the Mayer-Vietoris sequence yields $H^3_P(R) \oplus H^3_Q(R) \to H^3_J(R) \to H^4_m(R) \to H^4_P(R) \oplus H^4_Q(R)$. The first and last terms displayed are 0, since both P and Q have only two generators, and so the middle two terms are isomorphic. Thus, $H^3_I(R) \cong H^3_J(R) \cong H^4_m(R)$, as required. $\square$

5. If $n = 1$ the type is 1. Assume $n \geq 2$. A prime not containing x_i must contain all the other variables: let P_i be the prime generated by all the variables except x_i. Then it follows that $P_1, \ldots, P_n$ are all the minimal primes of R, and $R/P_i \cong K[[x_i]]$. Thus, $\dim\,(R) = 1$. Then $x = \sum_{i=1}^n x_i$ is a one element system of parameters: in fact, $x_i^2 = x_i x \in Rx$. Then R/Rx is $K \oplus V$ where V is spanned by over K by the x_i and has dimension $n - 1$ because $\sum_{i=1}^n x_i$ is 0 in R/Rx. Then V is the socle in R/Rx, and so the type of R is $n - 1$. $\square$

6. By a class theorem, we may take $\omega = \mathrm{Ext}^2_S(S/I,\, S)$ as a global canonical module for $R = S/I$, and the given resolution may be used to calculate it. We apply $\mathrm{Hom}_S(\_\,,\, S)$. If we use dual bases for the last two free modules, we obtain that $\omega = \mathrm{Ext}^2_S(S/I,\, S)$ is the cokernel of $X^* : S^3 \to S^2$, where X^* denotes the transpose of X. Map S^2 onto (x_{11}, x_{12}) by sending $(1, 0) \mapsto x_{12}$ and $(0, 1) \mapsto -x_{11}$. Then the image of X^* vanishes under the composite map: the columns of X^* map to 0 along with two of the minors, and all three are all 0 in R. This yields a surjection of the cokernel $\mathrm{Ext}^2_S(S/I,\, S) \twoheadrightarrow (x_{11},\, x_{12})R$. Since both modules are rank one torsion-free over R (the canonical module is because this holds locally, by a class theorem), the kernel must be 0, and the map is an isomorphism. $\square$

EXTRA CREDIT 5. Kill the the elements specified. The image of the matrix has entries equal to 0 below the first diagonal, a constant value x_i (the image of, say, $x_{1,i}$) on the i th diagonal, where $\underline{x} = x_1, \ldots, x_{s-r+1}$ are indeterminates, and entries equal to 0 above the last diagonal. Since the proposed system of parameters has the correct cardinality, it will suffice for both (a) and (b) to show the image of the ideal P is $(\underline{x})^r$, and it is clearly contained in this ideal. We use induction on both r and s. Each $r \times r$ minor involving the first column is x_1 times an $r - 1$ size minor of the matrix obtained by omitting the first row and column. The smaller matrix has the same form, and it follows that its $r - 1$ size minors generate the ideal $(\underline{x})^{r-1}$. This implies that the image of P contains all monomials of degree r in $\underline{x}$ that are divisible by x_1. Now suppose we have shown that the image of P contains all monomials of degree r in $\underline{x}$ that are divisible by any of the elements $x_1, \ldots, x_i$. To show that we get all monomials divisible by x_{i+1}, we first consider the problem modulo $x_1, \ldots, x_i$. Kill these variables, and omit the first i columns of the matrix. Then it follows by induction that the image of P contains the required monomials mod $x_1, \ldots, x_i$. Thus, each monomial of degree r divisible by x_{i+1} differs from a linear combination of elements of P by a sum S of multiples of the elements $x_1, \ldots, x_i$, and the coefficients can be taken of degree $r - 1$. Thus, S is a linear combination of monomials of degree r divisible by at least one of $x_1, \ldots, x_i$, and so is in the image of P. It follows that the image of P also contains all degree r monomials divisible by x_{i+1} as well. $\square$

(c) The type will be the same as the type of $K[\underline{x}]/(\underline{x})^r$, which is the number of monomials of degree $r - 1$ in $s - r + 1$ variables, or $\binom{s-r+1+(r-1)-1}{r-1} = \binom{s-1}{r-1}$.