Math 615, Winter 2011 **Problem Set #5**
Due: Monday, April 25

1. Let R be a ring of prime characteristic $p > 0$. Let S denote R viewed as an R-module by the Frobenius endomorphism F. If $R \to S$ splits as a map of R-modules, then R is called *F-split.*

(a) Show that a polynomial ring $R = K[x_1, \ldots, x_n]$ over a field K of positive characteristic p is F-split.

(b) Show that if R is F-split, then the action of F on $H^i_I(R)$ is injective for every ideal I of R.

(c) Suppose that R is a finitely generated K-algebra, where K is a field of characteristic $p > 0$, that R is F-split, and that R is $\mathbb{N}$-graded with $R_0 = K$. Let $m = \bigoplus_{k=1}^{\infty}[R]_k$ denote the homogeneous maximal ideal of R. Show that $[H^i_m(R)]_k = 0$ for all i and all $k > 0$.

2. Let K be a field of positive characteristic p, and let Δ be a finite simplicial complex with vertices $x_1, \ldots, x_n$. Let $R = K[x_1, \ldots, x_n]/I_\Delta$ be the Stanley-Reisner ring. Prove that R is F-split.

3. Assume the same hypothesis as in **1.** (c). Also assume that K is perfect, that R is a domain, and that R_P is Cohen-Macaulay for $P \neq m$. Let S denote R viewed as an R-module by the Frobenius endomorphism F, so that $S = R \oplus M$ as R-modules. Prove that M is Cohen-Macaulay.

4. Let (R, m, K) be a Gorenstein local ring and let T be a module-finite extension ring such that T is a Cohen-Macaulay local ring. Let $x_1, \ldots, x_n$ be a system of parameters for R, and let $I = (x_1, \ldots, x_n)R$. Show that $R \to T$ splits if and only if $IT \cap R = I$.

5. Let $(R, m\, K)$ be a local ring, let $X = \mathrm{Spec}\,(R)$, and let $Y = X - \{m\}$, the *punctured spectrum* of R.

(a) Show that Y is disconnected if and only if there exist ideals I, J in R such that $I + J$ is primary to m, $I \cap J$ consists of nilpotents, but neither I nor J consists of nilpotents.

(b) Prove that if Y is disconnected, then $H^1_m(R) \neq 0$. [The Mayer-Vietoris sequence may be helpful.] Hence, if $\mathrm{depth}_m R \geq 2$ then Y is connected.

6. Let $f(x)$ denote a transcendental power series (over $\mathbb{C}(x)$) in the maximal ideal of $\mathbb{C}[[x]]$, e.g., $e^x - 1$ or $\sin(x)$. Suppose that $f(x) = \sum_{i=1}^{\infty} a_i x^i$, and let $f_n(x) = \sum_{i=1}^{n} a_i x^i \in \mathbb{C}[x]$. Let $R = \mathbb{C}[x, y]_{\mathcal{M}}$ be a localized polynomial ring in two variables, where $\mathcal{M} = (x, y)$, let $m = \mathcal{M}R$, and let $I_n = (y - f_n, x^{n+1}) \subseteq R$. Show that I_n is a decreasing sequence of ideals primary to the maximal ideal m of R whose intersection is (0), but for all n, $I_n \not\subseteq m^2$. This shows that Chevalley's Lemma fails when R is not complete.

EXTRA CREDIT 6. Let S be the $\mathbb{N}$-graded ring defined in Problem **6.** of Problem Set #3. S has Krull dimension 3 and is a normal domain that is not Cohen-Macaulay. Let m be the homogeneous maximal ideal of S and let $E = E_S(S/m)$. Find an explicit homogeneous ideal I of S such that $\mathrm{Hom}_S(I, E) \cong H^3_m(S)$.