Math 615, Winter 2011 **Problem Set #5: Solutions**

1. (a) Since K is a field, the map $F : K \to K$ splits over K: note that $F(K)$ may be a proper subfield of K. Let θ be a splitting of this map of K-vector spaces. Extend θ to Θ on the polynomial ring R as follows: if $c \in K$ and μ is a monomial in the variables, $c\mu$ maps to 0 if some exponent in μ is not divisible by p, and if $c\mu = c\nu^p$ is such that every exponent in μ is divisible by p, the $c\mu$ maps to $\theta(c)\nu$. Since R is the direct sum of the abelian subgroups $K\mu$, this map is additive. By the distributive law, to check R-linearity, it suffices to check that $\Theta\big((c_1\lambda)^p(c\mu)\big) = (c_1\lambda)\Theta(\mu)$. Both are 0 if some exponent in μ is not divisible by p, and if $\mu = \nu^p$ both are $c_1\theta(c)\lambda\nu$, since $\theta(c_1^p c) = c_1\theta(c)$. Moreover, $\Theta\big((c\nu)^p\big) = c\nu$. $\square$

(b) Let S denote R as an algebra over itself with structural homomorphism F. Then $S = R \oplus M$ for some R-module M. Now $R \to S$ induces $H^i_I(R) \to H^i_I(S) = H^i_I(R) \oplus H^i_I(M)$, so that $H^i_I(R) \to H^i_I(S)$ is injective (even R-split), and this map is the action of Frobenius once one notes $H^i_I(S) = H^i_{IS}(S)$ and then remembers that S is R as a ring, so that the latter is $H^i_{I^{[p]}}(R) \cong H^i_I(R)$, since $I^{[p]}$ and I have the same radical. $\square$

(c) Let $H = H^i_m(R)$. We know that $[H]_s = 0$ for all $s > N$, where N is some sufficiently large positive integer, simply because H has DCC. The action of F maps $[H]_k$ into H_{pk} and this map is injective. Given $k > 0$, choose e such that $p^e k > N$. Then the e th iteration F^e of F maps $[H]_k$ injectively into $[H]_{p^e k} = 0$, and so $[H]_k = 0$. $\square$

2. The construction of the splitting is the same as in the polynomial ring case. The set of nonzero monomials is typically smaller, but this does not affect the argument once one notes that if μ, ν are monomials in the polynomial ring, then μ is 0 in $K[x_1, \ldots, x_n]/I_\Delta$ iff μ^p is 0, and $\mu\nu^p = 0$ in $K[x_1, \ldots, x_n]/I_\Delta$ iff $\mu\nu = 0$. (Both statements follow from the fact that the generators of I_Δ are square-free.) $\square$

3. By Problem **3.** of Problem Set #3, $H^i_m(R)$ has finite length for $i < n = \dim(R)$. Since S is isomorphic to R, $H^i_m(S)$ has the same length over the residue field of S, and since K is perfect, the map of residue fields is an isomorphism, and so $H^i_m(S)$ has the same length as $H^i_m(R)$ with both of them considered as R-modules. By **2.** (c) above, $H^i_m(S) \cong H^i_m(R)$ as R-modules. It follows that $H^i_m(M)$ has 0 length, and so is 0, for $i < n$. This shows that $\mathrm{depth}_m M = n$, and hence that M is Cohen-Macaulay. $\square$

4. Let $I_t = (x_1^t, \ldots, x_n^t)R$. By a class theorem, it suffices to show that $R/I_t \to T/I_tT$ is injective for all t (equivalently, that $I_tT \cap R = I_t$ for all t), since the ideals I_t form a descending sequence of irreducible ideals cofinal with the powers of m in the Gorenstein local ring R. Let $x = x_1 \cdots x_n$. Consider the commutative diagram:

$$\begin{array}{ccc} R/I & \xrightarrow{x^{t-1}} & R/I_t \\ {\scriptstyle g}\downarrow & & {\scriptstyle h}\downarrow \\ T/IT & \xrightarrow{x^{t-1}} & T/I_tT \end{array}.$$

Note that $I = I_1$. The horizontal rows are part of the direct limit systems used to calculate $H^n_I(R)$ and $H^n_{IT}(T)$, and g, h are induced by $R \hookrightarrow T$. Since R and T are Cohen-Macaulay, $x_1, \ldots, x_n$ is a regular sequence in each ring, and so the horizontal maps are injective, by a class theorem. Since g is given to be injective, the induced map $R/I \to T/I_tT$ is injective, and a socle generator in R/I has nonzero image in T/I_tT. But R/I_t has a one-dimensional socle as well, and so a socle generator in R/I must map to a socle generator in R/I_t. Hence, h does not kill the socle in R/I_t, and, hence, injects the socle into T/I_tT. Since R/I_t is an essential extension of its socle, it follows that h is injective. $\square$

5. (a) Disconnecting Y is equivalent to giving closed sets $V(I) \cap Y$ and $V(J) \cap Y$ that are proper, whose union is Y, and which are disjoint. Since every nonempty closed set contains m, the first condition is that $V(I), V(J)$ are proper in X, which is equivalent to the condition that neither I nor J consists entirely of nilpotents. Since the union is Y, $V(I \cap J) \supseteq Y$, which implies that $I \cap J$ consists of nilpotents. Disjointness means that $V(I + J) \subseteq \{m\}$. It cannot be empty, or else I or J is R, in which case the other must consist of nilpotents. Hence, $V(I + J) = \{m\}$, which means that $I + J$ is m-primary. $\square$

(b) Part of the Mayer-Vietoris sequence is $H^0_I(R) \oplus H^0_J(R) \xrightarrow{\alpha} H^0_{I\cap J}(R) \xrightarrow{\beta} H^1_{I+J}(R)$. Since I does not consist entirely of nilpotents, $H^0_I(R) \subseteq m$, and, similarly, $H^0_J(R) \subseteq m$. Since $I \cap J$ consists entirely of nilpotents, $H^0_{I\cap J}(R) = R$. It follows that the image of α is not all of R, and so β induces an injection of the nonzero cokernel into $H^1_{I+J}(R) = H^1_m(R)$. Hence, $H^1_m(R) \neq 0$, which implies that the depth of R on m at most 1. $\square$

6. Clearly, $\mathrm{Rad}\,(I_n)$ contains x, hence f_n and y as well, and so all of the I_n are m-primary. We have that $I_{n+1} \subseteq I_n$ for all n since $f_{n+1} - f_n$ is a multiple of x^{n+1}. Moreover, $y - f_n \notin m^2$. Let $S = \mathbb{C}[[x, y]]$. Note that if $J_n = I_nS$, then $J_n = (f, x^{n+1})S$. Since ideals in $\mathbb{C}[[x, y]]$ are mS-adically separated, the intersection of all the J_n is fS. Thus,to show that $\bigcap_{n=1}^{\infty} I_n = (0)$, it will suffice to show that $fS \cap R = 0$. Suppose that $G \in R - 0$ is in the intersection. We may multiply by an element of $R - m$ and so we may assume that $G \in \mathbb{C}[x, y] - 0$. There is a continuous $\mathbb{C}[[x]]$-homomorphism $\mathbb{C}[[x, y]] \twoheadrightarrow \mathbb{C}[[x]]$ that sends $y \mapsto f$. The kernel is a prime ideal containing $y - f$, and it follows that the kernel is precisely $(y - f)S$. But then $G(x, f) = 0$, contradicting the transcendence of f. $\square$

EXTRA CREDIT 6. Let $R = K[[xu, xv, yu, yv]] \cong K[[a, b, c, d]]/(ad - bc)$, which is Gorenstein. By local duality over R, $H^3_m(S)$ is the Matlis dual of $\mathrm{Hom}_R(S, R)$. Note that S is module-finite over R. This module is torsion-free over R and hence over S. Its torsion-free rank over R is the same as that of S. Hence it, is a rank one torsion-free S-module. As an R-module, $S = R \oplus (Ruz + Rvz) \oplus (Ru^2z^2 + Ruvz^2 + Rv^2z^2)$. Let $\_^*$ be $\mathrm{Hom}_R(\_, R)$. Hence, $S^* = R^* \oplus (Ruz + Rvz)^* \oplus (Ru^2z^2 + Ruvz^2 + Rv^2z^2)^*$. Each summand is rank one over the domain R, and so to map the second (resp., third) summand to R is equivalent to give two (respectively, three) elements of R with the same ratio as the generators: the map will be given by multiplication by a fraction. We obtain the following generators for S^*: f, which is the identity on R and kills the other two summands, g_1, g_2 which kill the first and third summands while mapping uz, vz to ux, vx or to uy, vy respectively, and three functionals h_1, h_2, h_3 that kill the first two summands and map u^2z^2, uvz^2, v^2z^2 to the respective elements obtained by replacing z^2 by x^2, by xy or by y^2, respectively. Then $(uz)g_1 = (ux)f$, $(uz)g_2 = (uy)f$, and $(u^2z^2)h_i$ is $u^2\mu_i f$ where μ_i is one of x^2, xy, y^2. Multiplying all six generators by u^2z^2 and taking the S-span therefore yields If where $I = (u^2z^2, (ux)(uz), (uy)(uz), u^2x^2, u^2xy, u^2y^2) = (ux, uy, uz)^2$. I is the required ideal. One may also use the ideal whose generators are obtained by multiplying any nonzero quadratic form in u and v by the elements x^2, y^2, z^2, xy, xz, yz.