D-MODULES AND LYUBEZNIK'S FINITENESS THEOREMS FOR LOCAL COHOMOLOGY

by Melvin Hochster

0. INTRODUCTION AND BACKGROUND

We shall restrict attention to the situation where I is an ideal of a Noetherian ring R. However, because some of the results we discuss apply to iterated local cohomology, we want to study the local cohomology modules $H^i_I(M)$ even when M need not be Noetherian.

We take as our definition that

$$H^i_I(M) = \varinjlim_t \mathrm{Ext}^i_R(R/I^t,\, M).$$

In particular,

$$H^0_I(M) = \varinjlim_t \mathrm{Hom}_R(R/I^t,\, M),$$

and this may be identified with

$$\bigcup_t \mathrm{Ann}_M I^t.$$

In these definitions the powers I^t may be replaced by any decreasing sequence of ideals cofinal with the powers of I. If $I = (f_1,\, \dots\,,\, f_h)R$, one may use, for example, the sequence $I_t = (f_1^t,\, \dots\,,\, f_h^t)R$. A mildly different alternative definition is to use the right derived functors of $H^0_I(\_)$: take a right injective resolution of M (omitting M), apply $H^0_I(\_)$, and then take cohomology.

The modules $H^0_I(M)$ only depend on the radical of I. They have the property that every element is killed by a power of I.

Of greater interest is the following alternative method of obtaining the local cohomolgy. Suppose that the radical of I is the same as the radical of $(f_1,\, \dots\,, f_h)R$. Then $H^i_I(M)$ is the same as the direct limit of the Koszul cohomology modules $\mathcal{K}^i(f_1^t,\, \dots\,,\, f_h^t; M)$. Yet

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another alternative is take the direct limit of Koszul complexes before taking cohomology. Using this idea one sees that $H^i_I(M)$ is the i th cohomology module of the complex

$$0 \to M \to \bigoplus_i M_{f_i} \to \bigoplus_{i_1<i_2} M_{f_{i_1} f_{i_2}} \to \cdots \to M_{f_1 \cdots f_h} \to 0.$$

This complex may be though of as the result of tensoring together the h complexes

$$0 \to R \to R_{f_i} \to 0$$

(where the copy of R is in degree zero and the copy of R_{f_i} in degree one), and then tensoring with M. Note that it follows that $H^i_I(M) = 0$ if i exceeds the number of genrators of I or of any ideal with the same radical as I.

Extending sections: let $X = \mathrm{Spec}\,(R)$, and let $U = X - V(I)$. Let $\mathcal{M}$ be the sheaf associated with the R-module M. Then there is a long exact sequence which begins:

$$0 \to H^0_I(M) \to H^0(X,\, \mathcal{M}) \to H^0(U,\, \mathcal{M}|_U) \to H^1_I(M) \to 0$$

and shows that $H^i_I(M) \cong H^{i-1}(U, \mathcal{M}|_U)$ for $i \geq 2$. We are using that, because X is affine, $H^i(X,\, \mathcal{M}) = 0$ for all $i \geq 1$. This shows that $H^1_I(M)$ represents the obstruction to extending a section of $\mathcal{M}|_U$ to all of X.

Any of the definitions shows that if one has a short exact sequence of modules

$$0 \to M' \to M \to M'' \to 0$$

then there is a long exact sequence for local cohomology:

$$H^0_I(M') \to H^0_I(M) \to H^0_I(M'') \to H^1_I(M') \to \cdots$$

$$\to H^i_I(M') \to H^i_I(M) \to H^i_I(M'') \to H^{i+1}_I(M') \to \cdots$$

Another consequence of the Koszul cohomology point of view is that if $R \to S$ is a ring homomorphism, I is an ideal of R, $J = IS$, and M is an S-module, which may also be viewed as an R-module by restriction of scalars, then $H^I_I(M) = H^i_{IS}(M)$. The complexes used to compute the two cohomology modules are the same.

If M is a finitely generated R-module and $I = IM$ then $I + \mathrm{Ann}_R M = R$, and it follows that all the modules $H^i_I(M) = 0$. If M is finitely generated and $IM \neq M$, then $H^i_I(M)$ vanishes for $i < d = \mathrm{depth}\,_I M$, but with $H^d_I(M) \neq 0$. Thus, the first nonvanishing local cohomology module of M is governed by the depth of M on I.

If $I = m$ is a maximal ideal of R and M is finitely generated we have that $H^i_m(M)$ does not change if we replace m by mR_m and M by M_m. We may also complete without changing anything. The local cohomology has DCC in this case.

In general, $H^i_I(M)$ does not have ACC nor DCC, even when M is finitely generated. Typically, it is a huge module.

What about the set of associated primes of a local cohomology module? Is it necessarily finite? The answer is no. But it is an open question whether the set of minimal primes of $H^i_I(M)$ is finite when M is Noetherian. This question is open even when M is R and R is local.

Suppose that (R, m, K) is local. Must the socle, i.e., the annihilator of m, in a local cohomology module $H^i_I(M)$ be finite if M is finitely generated? This is not true even if R is local and $M = R$, although it was conjectured by Grothendieck. However, if $R = K[[x, y, u, v]]/(xu-yv)$ then the socle in $H^2_{(x,y)}(R)$ is an infinite-dimensional K-vector space.

I want to survey some results in the literature about such matters. In some cases details will be given. In others, details would take a semester or a year and I will only sketch ideas. For example, we shall make use of the theory of D-modules in equal characteristic 0, but I will not do all details of this theory.

1. D-MODULES

This is an expository sketch of part of the theory of D-modules and their application to the proof of some of the results of Lyubeznik [Lyub1] on local cohomology over regular rings in equal characteristic zero. Our objective is to give a treatment of several of the less technical results that will give an indication of the methods. We want to note that the results of [Lyub1] are stronger than those obtained here in that the class of functors T allowed in results like Theorem (6.2) is larger, and that, moreover, the results proved here for $T(R)$ hold, in the case where R is a formal power series ring over a field of characteristic 0, when $T(R)$ is replaced by any holonomic $D(R,\, K)$-module.

The main source for the treatment of D-modules here is Jan-Erik Björk's book *Rings of Differential Operators*, which we refer to as [Björk]. Most of the proofs of results from Björk's book have been omitted, but, in some cases, there is an indication of the key idea.

Similar results are obtained in positive prime characteristic p by Frobenius methods in [HunSh] and [Lyub2].

I want to thank Toby Stafford for several valuable conversations, and for pointing out the reference [Gab-Lev].

1. The basic set-up: filtered rings. Throughout we shall be considering an associative (but not necessarily commutative) ring A with 1 equipped with an *ascending* filtration

$$\Sigma_0 \subseteq \Sigma_1 \subseteq \Sigma_2 \subseteq \cdots$$

(this means that each Σ_i is an additive subgroup, that $1 \in \Sigma_0$, that $\bigcup_i \Sigma_i = A$, and that for all i and j we have that $\Sigma_i\Sigma_j \subseteq \Sigma_{i+j}$). There is an obvious way of putting a ring structure on $\mathrm{gr}(A) = \Sigma_0 \oplus \Sigma_1/\Sigma_0 \oplus \cdots \oplus \Sigma_i/\Sigma_{i-1} \oplus \cdots$ and we shall assume throughout that A has a filtration such that $\mathrm{gr}(A)$ is *commutative* and *Noetherian*, although we usually repeat this hypothesis when stating theorems. Modules are assumed to be left modules unless otherwise specified. We shall also assume frequently, but not always, that $\mathrm{gr}(A)$ is a regular commutative Noetherian ring.

2. The main example: $\boldsymbol{D(R,\,K)}$. Let K be a field of characteristic 0 and let R denote the formal power series ring $K[[x_1, \dots, x_n]]$ in n variables over K, where $n \geq 0$ is an integer. Let $D = D(R,\,K)$ denote the subring of the K-vector space endomorphisms of R generated by R (each element of R yields an endomorphism of itself via multiplication by that element) and the usual differential operators $\delta_1, \dots, \delta_n$, defined formally, so that $\delta_i f = \frac{\partial f}{\partial x_i}$. Then we may write $D(R,\,K) = R\langle \delta_1, \dots, \delta_n\rangle$, where $\langle\ \rangle$ indicate adjunction of elements to a ring in the non-commutative case. This is very similar to the definition of the Weyl algebra (the only difference is that in the Weyl algebra one uses the polynomial ring $K[x_1, \dots, x_n]$ in place of the formal power series ring R). Then $D(R,\,K)$ has a filtration in which Σ_i consists of all R-linear combinations of monomials in the elements δ_j such that the monomial has degree at most i in the δ_j. Note that

$$\delta_i\delta_j - \delta_j\delta_i = 0$$

for all i, j, that

$$\delta_i x_j - x_j\delta_i = 0$$

if $j \neq i$ and that

$$\delta_i x_i - x_i\delta_i = 1$$

(as operators on R) (this is a consequence of the rule for differentiating a product). More generally, for any $f \in R$ and every i one has the relation

$$\delta_i f - f\delta_i = \frac{\partial f}{\partial x_i}.$$

It follows that every element of $D(R,\,K)$ can be written as an R-linear combination of monomials in the δ_i (the latter mutually commute), i.e., that $D(R,\,K)$ is R-free on these monomials as a left R-module, and it is also easy to check that it is R-free on these monomials as as a right R-module. It then follows easily as well that $\mathrm{gr}(D(R,\,K))$ is isomorphic with a polynomial ring $R[\zeta_1, \dots, \zeta_n]$ in n variables over R, and so is commutative, Noetherian, and regular. (Here, ζ_i is the image of δ_i in Σ_1/Σ_0.) This ring has Krull dimension $2n$.

R is a module over $D(R,\,K)$, but since we also think of elements of R as in $D(R,\,K)$, an expression such as $\delta_i f$ with $f \in R$ has two possible meanings: if this is δ_i acting on an element of the $D(R,\,K)$-module R, the value is $\frac{\partial f}{\partial x_i} \in R$. But if the multiplication is being performed in $D(R,\, K)$ the value is the operator that sends $g \in R$ to $\frac{\partial(fg)}{\partial x_i} = \frac{f\partial g}{\partial x_i} + \frac{\partial f}{\partial x_i}g$, and this operator is the same as $f\delta_i + \frac{\partial f}{\partial x_i}$, which is the reason for the third displayed relation.

It is not true, as asserted on p. 100 of [Björk], that all maximal ideals of $\mathrm{gr}(D(R,\, K)) \cong K[[x_1,\, \dots\, ,x_n]][\zeta_1,\, \dots\, ,\zeta_n]$ have height $2n$: e.g., if $n = 1$, $x_1\zeta_1 - 1$ generates a maximal ideal of height 1, and there are, in general, maximal ideals of height $2n-1$ as well as $2n$. This error is inessential: the hypothesis that $\mathrm{gr}(A)$ be regular of pure dimension frequently assumed in [Björk] can be replaced throughout by the weaker hypothesis that the height of all homogeneous maximal ideals be the same, since the annihilator and associated primes of graded modules over $\mathrm{gr}(A)$ will be homogeneous, and the Krull dimension of an $\mathbb{N}$-graded commutative Noetherian ring is always the same as the supremum of the heights of its homogeneous maximal ideals. Following the appendix to the unpublished manuscript [Gab-Lev], we shall say that an $\mathbb{N}$-graded commutative Noetherian ring has *pure graded dimension* h if every homogeneous maximal ideal has height h, and then h is the same as the Krull dimension of the ring. The results that are proved in [Björk] under the hypothesis that $\mathrm{gr}(A)$ be commutative, Noetherian and regular of pure dimension actually hold when $\mathrm{gr}(A)$ is commutative, Noetherian, and regular of pure graded dimension, and so may be applied to $D(R,\, K)$.

3. Good filtrations of modules and dimension. If A is filtered with $\mathrm{gr}(A)$ commutative Noetherian as in §1. then every finitely generated (left) A-module M has a *good* filtration Γ, i.e., an ascending filtration, indexed by $\mathbb{Z}$, consisting of abelian subgroups $\{\Gamma_i\}$ such that $\Gamma_i = 0$ for all sufficiently small values of i, such that $\bigcup_i \Gamma_i = M$, such that $\Sigma_i\Gamma_j \subseteq \Gamma_{i+j}$ for all $i,\, j \in \mathbb{N}$ (the conditions so far are the conditions for a filtration) and such that the associated graded module

$$\mathrm{gr}_\Gamma(M) = \bigoplus_{i=1}^{\infty} \Gamma_i/\Gamma_{i-1}$$

is finitely generated over $\mathrm{gr}(A)$ (this last condition is the condition for the filtration to be *good*).

(3.1) Proposition. *Let A be a filtered ring as above such that* $\mathrm{gr}(A)$ *is commutative and Noetherian. Then A is both left and right Noetherian. Moreover, an A-module M has a good filtration if and only if it is finitely generated as an A-module, and if $\{\Gamma_i\}$ and $\{\Gamma'_i\}$ are two such filtrations then there exists an integer c such that $\Gamma_i \subseteq \Gamma'_{i+c}$ and $\Gamma'_i \subseteq \Gamma_{i+c}$ for all i.*

(This is Proposition (6.1) on p. 69 of [Björk].)

When M is a finitely generated A-module we shall write $d(M)$ for the Krull dimension of the module $\mathrm{gr}_\Gamma(M)$ (in the commutative rings sense). We refer to $d(M)$ as the *dimension* of M. The comparison of filtrations in Proposition (3.1) can be used to show that $d(M)$ is independent of the choice of Γ. See Lemma (6.2) on p. 69 of [Björk]. Once one knows this one has:

(3.2) Proposition. *Let A be a filtered ring as above such that* $\mathrm{gr}(A)$ *is commutative and Noetherian. Given a short exact sequence of nonzero A-modules* $0 \to M_1 \to M_2 \to M_3 \to 0$ *such that M_2 is finitely generated (this holds if and only if both M_1 and M_3 are finitely generated), one has* $d(M_2) = max\{d(M_1),\, d(M_3)\}$.

This is Lemma (6.2) of [Björk]. The point is that if one has a good filtration of M_2, it induces a good filtration on M_1 (by intersecting) and on M_3 (by taking images), and then one has a short exact sequence $0 \to \mathrm{gr}(M_1) \to \mathrm{gr}(M_2) \to \mathrm{gr}(M_3) \to 0$, and the result follows.

4. The regular case. We next focus on the case where $\mathrm{gr}(A)$ is assumed to be regular, of finite Krull dimension, as well as commutative and Noetherian.

We first recall that the *weak global dimension*, denoted $\mathrm{w.gl.dim}(A)$, of a ring A is d if whenever M is a right A-module and N is a left A-module then $\mathrm{Tor}_i^A(M,\, N) = 0$ for $i > d$ while there exist $M,\, N$ such that $\mathrm{Tor}_d^A(M,\, N) \neq 0$. For a regular Noetherian commutative ring S, the weak global dimension of S is the same as the Krull dimension of S. The following result, which holds without the commutative Noetherian restrictions on $\mathrm{gr}(A)$, is Theorem (3.7) of [Björk]:

(4.1) Theorem. *If A is any filtered ring,* $\mathrm{w.gl.dim}(A) \le \mathrm{w.gl.dim}(\mathrm{gr}(A))$.

Of course, this is not interesting unless $\mathrm{w.gl.dim}(\mathrm{gr}(A))$ is finite.

For the rest of this section we shall assume that A is a filtered ring such that $\mathrm{gr}(A)$ is commutative, Noetherian, and regular of finite pure graded dimension, which will then be the same as $\mathrm{w.gl.dim}(\mathrm{gr}(A))$. If M is a finitely generated A-module we denote by $j(M)$ the smallest nonnegative integer j such that $\mathrm{Ext}_A^j(M,\, A) \neq 0$.

The following is Theorem (7.1) of [Björk].

(4.2) Theorem. *Let A be a filtered ring such that* $\mathrm{gr}(A)$ *is commutative, Noetherian, and regular of finite pure graded dimension. Then for every finitely generated A-module M,* $d(M) + j(M) = \mathrm{w.gl.dim}(\mathrm{gr}(A))$.

From this one deduces:

(4.3) Corollary. *With A as in (4.2), for every finitely generated A-module M we have that* $d(M) \ge \mathrm{w.gl.dim}(\mathrm{gr}(A)) - \mathrm{w.gl.dim}(A)$.

The finitely generated modules M such that $d(M) = \mathrm{w.gl.dim}(\mathrm{gr}(A)) - \mathrm{w.gl.dim}(A)$, together with 0, constitute the (left) Bernstein class. Since they are of smallest possible dimension, their submodules and quotient modules are also in the Bernstein class, and if $0 \to M_1 \to M_2 \to M_3 \to 0$ is exact then M_2 is in the Bernstein class if and only if both M_1 and M_3 are in the Bernstein class. In the case where $A = D(R,\, K)$ as in §2, we shall refer to the modules in the Bernstein class as *holonomic.*

There is an analogous notion for right modules. This is important, because it turns out that one has:

(4.4) Theorem. *With A as in (4.2) and with $\mu = \mathrm{w.gl.dim}(A)$, the functor that sends M to $\mathrm{Ext}^{\mu}_A(M,\, A)$ is an exact contravariant functor that interchanges the left Bernstein class and the right Bernstein class. Moreover, $M \cong \mathrm{Ext}^{\mu}_A(\mathrm{Ext}^{\mu}_A(M,\, A),\, A)$ naturally if M is in either the left or the right Bernstein class, so that one has an anti-equivalence of categories. In consequence, the modules in the Bernstein class have finite length as A-modules, i.e., each nonzero module in the Bernstein class has a finite filtration by simple A-modules (each of which is again in the Bernstein class).*

See Definition (7.12), Theorem (7.13) and Lemma (7.14) on p. 76 of [Björk]. Note that to show that M has finite length one wants to prove that M has both ACC (this is automatic here) and that M has DCC. (That M has DCC enables one to choose a nonzero simple submodule $M_1 \subseteq M$ to begin the filtration, and then one does this again for M/M_1, etc. Since M has ACC the process terminates.) One uses the fact that the dual (right) module $\mathrm{Ext}^{\mu}_A(M,\, A)$ has ACC as a right A-module to prove that M has DCC as a left module.

5. Back to the main example: holonomic modules and local cohomology.

Throughout this section K is a field of characteristic 0, $R = K[[x_1, \dots, x_n]]$ for some fixed integer $n \geq 0$, and $D(R,\, K) = R\langle \delta_1, \dots, \delta_n\rangle$ as in §2. We want to apply the results of the preceding section with $A = D(R,\, K)$ as in §2. Then $\mathrm{w.gl.dim}(\mathrm{gr}(A)) = 2n$. With much more difficulty one can prove:

(5.1) Theorem. *Let K be a field of characteristic 0, let $R = K[[x_1, \dots, x_n]]$ be the formal power series ring and let $D(R,\, K) = R\langle \delta_1, \dots, \delta_n\rangle$. Then $\mathrm{w.gl.dim}(D(R,\, K)) = n$.*

This is Proposition (1.8) on p. 90 of [Björk]. We shall call a $D(R,\, K)$-module M *holonomic* if it is in the Bernstein class, which in this case means that M is finitely generated over $D(R,\, K)$ and that $d(M) = n$, since $\mathrm{w.gl.dim}\big(\mathrm{gr}(D(R,\, K))\big) - \mathrm{w.gl.dim}(D(R, K)) = 2n - n = n$.

We note that R itself, with the differential operators δ_i acting as usual, is a holonomic $D(R,\, K)$-module.

If W is any multiplicative system in $D(R,\, K)$ and M is a $D(R,\, K)$-module, then $W^{-1}M$ has the structure of a $D(R,\, K)$-module in such a way that the map $M \to W^{-1}M$ is a homomorphism of $D(R, K)$-modules. The issue is to extend the actions of the δ_i to the localized module. One extends the operator δ_i by the usual quotient rule for differentiation, $\delta_i(m/f) = (f \cdot \delta_i m - \frac{\partial f}{\partial x_i} \cdot m)/f^2$.

Of critical importance for applications to local cohomology theory is the following difficult result, which is Theorem (4.1) on p. 113 of [Björk].

(5.2) Theorem. *With K, R, n, and $D(R,\, K)$ as in (5.1), if M is a holonomic $D(R,\, K)$-module and $f \in R$, then the localization M_f is a holonomic $D(R,\, K)$-module.*

(5.3) Corollary. *With K, R, n, and $D(R, K)$ as in (5.1), let M be a module over $D(R, K)$. Let I be an ideal of R. Then the local cohomology modules $H^j_I(M)$ all have the structure of $D(R, K)$-modules in such a way that if M is holonomic, then all the $H^j_I(M)$ are holonomic.*

Proof. Let $I = (f_1, \dots, f_s)R$. Then $H^i_I(M)$ may be described as the ith cohomology module of the complex

$$0 \to M \to \bigoplus_i M_{f_i} \to \bigoplus_{i<j} M_{f_i f_j} \to \cdots \to M_{f_1 \cdots f_s} \to 0$$

where the maps are constructed from the various natural maps $M_g \to M_{gf_i}$, possibly with a sign, where g is a product of a subset of the f 's. It follows that each term in this complex is a $D(R, K)$-module and is holonomic if M is, and the maps are $D(R, K)$-linear. Thus, each kernel and each image is a $D(R, K)$-module and is holonomic if M is, and the same holds for the cohomology of the complex. □

(5.4) Corollary. *With K, R, n, and $D(R, K)$ as in (5.1), if M is a simple $D(R, K)$-module then the assassinator of M as an R-module contains a unique prime P of R. Hence, if M is a holonomic $D(R, K)$-module, then the assassinator of M as an R-module is finite.*

Proof. If M is simple and P is an associated prime, then $H^0_P(M)$ is nonzero and is a submodule of M, hence, all of M. If Q is another associated prime we also get $H^0_Q(M) = M$, and it follows that $P = Q$. The final statement follows from the fact that a holonomic module has a finite filtration by simple holonomic modules. □

By an obvious induction we also have:

(5.5) Corollary. *With K, R, n, and $D(R, K)$ as in (5.1), if $I_1, \dots, I_s$ is a sequence of ideals of R, $i_1, \dots, i_s$ is a sequence of integers, and M is a holonomic $D(R, K)$-module, then $H^{i_s}_{I_s}\big(H^{i_{s-1}}_{I_{s-1}}(\cdots H^{i_1}_{I_1}(M)\cdots)\big)$ is a holonomic $D(R, K)$-module. In particular, $H^{i_s}_{I_s}\big(H^{i_{s-1}}_{I_{s-1}}(\cdots H^{i_1}_{I_1}(R)\cdots)\big)$ is a holonomic $D(R, K)$-module, and so has a finite assassinator.* □

(5.6) Proposition. *With K, R, n, and $D(R, K)$ as in (5.1), let m be the maximal ideal of R. Then $D(R, K)/D(R, K)m \cong H^n_m(R)$ as an R-module, and this, of course, may be thought of as $E_R(R/m)$.*

Proof. Let S be the polynomial ring $K[x_1, \dots, x_n]$ and let Q be the ideal $(x_1, \dots, x_n)S$. Then $H^n_m(R) \cong H^n_{QS_Q}(S_Q) \cong H^n_Q(S)$. Let y_t denote the product of the x_j other than x_t, so that $y_t x_t = x_1 \cdots x_n$. Then $H^n_Q(S) \cong \operatorname{Coker}\big(\bigoplus_{t=1}^n S_{y_t} \to S_{x_1 \cdots x_n}\big)$, which is the R-span of the monomials $x_1^{-j_1} \cdots x_n^{-j_n}$ where $j_1, \dots, j_n$ are positive integers. Since $D(R, K)$ is R-free as a right R-module on the monomials in $\delta_1, \dots, \delta_n$, it follows that $D(R, K)/D(R, K)m$ is K-free on the span of the images of the monomials in the δ_i. There is a $D(R, K)$-linear map of $D(R, K)$ to $H^n_Q(S)$ that sends 1 to the image of $x_1^{-1} \cdots x_n^{-1}$. This maps kills m and hence the left ideal $D(R, K)m$, so that we have an induced map

$D(R, K)/D(R, K)m \to H^n_Q(S)$. The image of the element represented by $\delta_1^{i_1} \cdots \delta_n^{i_n}$ is represented by

$$\frac{\partial^{i_1}}{\partial x_1^{i_1}} \cdots \frac{\partial^{i_n}}{\partial x_n^{i_n}}(x_1^{-1} \cdots x_n^{-1}) = (-1)^{i_1+\cdots+i_n} i_1! \cdots i_n!\, x_1^{-(i_1+1)} \cdots x_n^{-(i_n+1)},$$

from which it follows that this $D(R, K)$-linear map carries a K-vector space basis for the domain module to a K-vector space basis for the target module, and so is an isomorphism. □

(5.7) Proposition. *With K, R, n, and $D(R, K)$ as in (5.1), let m be the maximal ideal of R. If M is any $D(R, K)$-module (no finiteness condition is necessary) such that every element is killed by a power of m, then M is isomorphic with a direct sum of copies of $D(R, K)/D(R, K)m$. If M is holonomic the direct sum is finite.*

Proof. M is an essential extension of a K-vector space $V \subseteq M$, and we may choose a K-vector space basis $\{v_\lambda\}_{\lambda\in\Lambda}$ for V. Consider a free $D(R, K)$-module G with a free basis $\{u_\lambda\}_{\lambda\in\Lambda}$ indexed by Λ. Since each copy of $D(R, K)$ is both a left and right module over $D(R, K)$, it follows that G is both a left and left right module over R. We map G to M by sending the free generator u_λ to the element v_λ. This induces a map $G/Gm \to M$, that sends the images of the u_λ to the corresponding elements in the basis v_λ for V. Note that G/Gm may be identified with the direct sum of a family of copies of $D(R, K)/D(R, K)m$ indexed by Λ. Thus, G/Gm is an essential extension of a K-vector space, its socle, that is mapped isomorphically onto V. Since the map $G/Gm \to M$ is injective on the socle in G/Gm, it is injective, and we know that G/Gm is injective as an R-module. Thus, M may be written as the direct sum of the image of G/Gm and another module, M_0. Since $M_0 \subseteq M$, every element is killed by a power of m. If $M_0 \neq 0$, then it has a nonzero socle. But this is a contradiction, since the entire socle V of M is contained in the image of G/Gm. Thus, $M_0 = 0$, and $G/Gm \cong M$, as required. □

For convenience we shall use the letter T to indicate a functor obtained by composition of a finite number of local cohomology functors H^i_I, where both i and I are allowed to vary, as in the statement of Corollary (5.5).

(5.8) Corollary. *With K, R, n, and $D(R, K)$ as in (5.1), let m be the maximal ideal of R. Let T be a composition of local cohomology functors as described just above. If $T(R)$ has the property that every element is killed by a power of m, then $T(R)$ is isomorphic as a $D(R, K)$-module with a finite direct sum of copies of $D(R, K)/D(R, K)m$, and so is injective.*

Proof. By Corollary (5.5), $T(R)$ is a holonomic $D(R, K)$-module and then the result is immediate from Propositions (5.7) and (5.6). □

6. Bass numbers and the main results for arbitrary equal characteristic 0 regular rings. Recall that if R is Noetherian and M is any R-module, then $\mu^i(P, M)$ denotes the ith Bass number of M with respect to P, i.e., the number of copies of the injective hull $E_R(R/P)$ of R/P over R occurring as direct summands in the i th injective module of a minimal injective resolution of M. With $\kappa(P) = R_P/PR_P$ (which is isomorphic with the fraction field of R/P), one also has $\mu^i(P, M) = \dim_{\kappa(P)}\mathrm{Ext}^i_{R_P}(\kappa(P), M_P)$.

We should mention that in [Björk], it is shown (lines 3–6 on p. 109) that for every holomorphic $D(R, K)$-module M, $\mu^0(P, M)$ is finite. (This is established in the course of proving quite a different result: that if M is a holonomic $D(R, K)$-module and ϕ is a $D(R, K)$-module endomorphism of M, then ϕ is *algebraic* over K in the sense that it satisfies a polynomial equation with coefficients in K. See Proposition (3.11) and Remark (3.12) on p. 103 of [Björk].) But this is rather difficult, and we shall not give the argument here.

However, in case the holonomic D-module M is actually iterated local cohomology of R, we can proceed as follows. Localize at P and then complete. The new regular local ring is power series over a larger field, but we have not changed the value of μ^0 for P in replacing P by the expansion of P to the completion of R_P. The new module is still an iterated local cohomology module of the ring. We therefore have reduced to the case where of calculating $\mu^0\big(m, T(R)\big)$ for an iterated local cohomology functor T. We may replace $T(R)$ by $H^0_m\big(T(R)\big)$. The result is now immediate from (5.8).

(6.1) Lemma. *Let R be any commutative ring, let M be an R-module, and suppose that $0 \to M \to E^0 \to E^1 \to \cdots \to E^i \to \cdots$ is a minimal injective resolution of M.*

(a) *Let m be a maximal ideal of R. Then for all $i \geq 0$, the map $\mathrm{Ann}_{E^i}m \to \mathrm{Ann}_{E^{i+1}}m$ is 0.*

(b) *Suppose that (R, m, K) is local Noetherian and M is an arbitrary R-module. Suppose that $H^i_m(M)$ is injective as an R-module for all i. Then the map $\alpha_i : H^0_m(E^i) \to H^0_m(E^{i+1})$ is 0 for all i. It follows that, for all i, $H^i_m(M) \cong E_R(K)^{\mu^i(m,M)}$ and that for all i, $\mu^i(m, M) = \mu^0\big(m, H^i_m(M)\big)$.*

Proof. (a) Since E^i is an essential extension of the image W of E^{i-1}, each element v of $\mathrm{Ann}_{E^i}m$ has a nonzero multiple in W, which must lie in $\mathrm{Ann}_W m$. But such a multiple must be the same as the product of v with a nonzero element of R/m, which has an inverse in R/m, and it follows that v itself is in $\mathrm{Ann}_W m$, i.e., v is in the image of E^{i-1} and so maps to 0 in E^{i+1}, as required.

For part (b), first note that the cohomology of the complex $H^0_m(E^\bullet)$ is $H^\bullet_m(M)$, and so consists of injective modules. Now, $E^j \cong \bigoplus_P E_R(R/P)^{\mu^j(P,M)}$ where P runs through all prime ideals of R and $E_R(R/P)^{\mu^j(P,M)}$ denotes the direct sum of $\mu^j(P,M)$ copies of $E_R(R/P)$. Since the only associated prime of $E_R(R/P)$ is P, it follows that $H^0_m(E^j) \cong E_R(K)^{\mu^j(m,M)}$.

Suppose the result is false, and let i be the least integer such that $\alpha_i \neq 0$. Then the image of $H^0_m(E^{i-1})$ in $H^0_m(E^i)$ is 0, and it follows that $H^i\big(H^0_m(E^\bullet)\big) \cong H^i_m(M)$ is the kernel W of α_i. The hypothesis implies that W is an injective module, and so splits from $H^0_m(E^i)$, say $H^0_m(E^i) = W \oplus_R W'$, where we think of the direct sum as internal. By part (a), W contains the entire socle of $H^0_m(E^i)$, and this implies that $W' \subseteq H^0_m(E^i)$ is zero, for a nonzero module in which every element is killed by a power of m must have a nonzero socle. But then $W = H^0_m(E^i)$, which implies that $\alpha_i = 0$, contradicting our assumption.

The statements in the final sentence now follow from our earlier observation that $H^0_m(E^j) \cong E_R(K)^{\mu^j(m,M)}$ together with the fact that all the maps vanish. $\square$

The following results may now be obtained for an arbitrary regular ring of equal characteristic zero. As mentioned earlier, we shall use T to denote the composition of several local cohomology functors H^i_I where i and I may vary.

(6.2) Theorem. *Let R be any regular Noetherian of equal characteristic 0. If T is a composition of local cohomology functors as described just above then:*

(a) *For any given maximal ideal of m, the set of associated primes of $T(R)$ contained in m is finite.*

(b) *If the assassinator of $T(R)$ consists entirely of maximal ideals of R then $T(R)$ is injective; if, moreover, (R, m, K) is local, then $T(R)$ is a finite direct sum of copies of $E_R(K)$.*

(c) *For every prime ideal P of R, all the Bass numbers of $T(R)$ for P are finite.*

(d) *The injective dimension of $T(R)$ (i.e., the length of a minimal injective resolution of $T(R)$) is bounded by the $h = \sup\{\dim R/P : P \in \mathrm{Ass}\,(T(R))\}$.*

Proof. (a) Suppose that the assassinator of $T(R)$ contains an infinite sequence of distinct primes $P_1, \ldots, P_s, \ldots$ within m. Then we can replace R by R_m and still have a counterexample: we may replace the ideals used in defining T by their expansions to R_m. We may similarly replace R by $\widehat{R}$: $\widehat{R} \otimes_R T(R) \cong T(\widehat{R})$, and we may again expand the ideals used in defining T. Moreover, if R/P embeds in $M = T(R)$, then $\widehat{R}/P\widehat{R}$ embeds in $\widehat{R} \otimes_R M$, and for each P_i we may choose a minimal prime of $P_i\widehat{R}$ lying over P_i and then $\widehat{R}/Q_i$ embeds in $\widehat{R}/P\widehat{R}$, so that Q_i will be an embedded prime of $\widehat{R} \otimes_R T(R)$. We therefore get a counterexample over the completion of R, which is isomorphic with formal power series over a field. Now the result is immediate from Corollary (5.5).

To prove (b) we note that the hypothesis on $T(M)$ implies that it is the direct sum of its submodules $H^0_m(T(M))$ as m varies through the maximal ideals of R. Thus, it suffices to see that the result holds when the assassinator consists of a single maximal ideal of R. But, as in part (a), the situation is unaffected if we first localize R at this maximal ideal, and then complete. The result is then immediate from Corollary (5.8).

(c) To see that the Bass numbers are finite we note that the Bass numbers for P do not change when we replace $T(R)$ by $T(R)_P \cong T(R_P)$, and in describing T we may replace each ideal that occurs as a subscript by its expansion to R_P. Thus, we may assume without loss of generality that (R, m, K) is local and that $P = m$. But if one of the Bass numbers is infinite we may replace R by its completion without affecting that, and, again, $\widehat{R} \otimes_R T(R) \cong T(\widehat{R})$. Moreover, as before, in the description of T we may replace each subscript ideal by its expansion to $\widehat{R}$. Therefore, we may assume without loss of generality that R is complete and that $P = m$. Thus, we may think of R as $K[[x_1, \ldots, x_n]]$. We first note that every $H^i_m(T(R))$ is a holonomic module, since $H^i_m \circ T = T'$ is a functor of the same form as T, and so isomorphic with a finite direct sum of copies of $E_R(K)$, by part (b). It follows from Lemma (6.1b) that $\mu^i(m, T(R)) = \mu^0(H^i_m(T(R))$ and the result follows.

(d) To prove the result it suffices to prove that $\mu_i(P, T(R)) = 0$ for all primes P and for all $i > h$. Assume the result false and and choose Q of smallest possible height so that $T(R_Q)$ gives a counterexample. Thus, we still have a counterexample after localization at Q, but we do not have a counterexample if we localize further. Note that when we localize further the supremum of the dimensions of the R/P for $P \in \operatorname{Ass} T(R)$ must decrease. Thus, we may assume that (R, m, k) is local and that $\mu^i(P, T(R)) = 0$ if $P \neq m$ and $i \geq h$ while $\mu^{h+1}(m, T(R)) \neq 0$ or the injective dimension will be h.

Thus, if $0 \to T(R) \to E^0 \to E^1 \to \cdots \to E^i \to \cdots$ is a minimal injective resolution of $T(R)$ we have that $E^i \cong H^0_m(E^i) \cong E_R(K)^{\mu_i(m, T(R))}$ for all $i \geq h$. It follows from Lemma (6.2b) that the maps $H^0_m(E^i) \to H^0_m(E^{i+1})$ vanish for all i, and, hence, that the maps $E^i \to E^{i+1}$ vanish for $i \geq h$. Since $E^\bullet$ is a minimal injective resolution, this can only happen if $E^{i+1} = 0$ for $i > h$, as required. $\square$

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