

Completion of rings and modules

Let (Λ, \leq) be a directed set. This is a partially ordered set in which for any two elements λ, μ there is an element ν such that $\lambda < \nu$ and $\mu < \nu$. Examples include any totally ordered set, e.g., the positive integers under \leq , and the finite subsets of a given set ordered by \subseteq . We want to describe inverse limits over Λ . An inverse limit system consists of objects X_λ indexed by Λ and for all $\lambda \leq \mu$ a morphism $f_{\lambda,\mu} : X_\mu \rightarrow X_\lambda$. A candidate for the inverse limit consists of an object X together with maps $g_\lambda : X \rightarrow X_\lambda$ such that for all $\lambda \leq \mu$, $g_\lambda = f_{\lambda,\mu} \circ g_\mu$. A candidate Y together with morphisms $h_\lambda : Y \rightarrow X_\lambda$ is an inverse limit precisely if for every candidate (X, g_λ) there is a unique morphism $k : X \rightarrow Y$ such that for all λ , $g_\lambda = h_\lambda \circ k$. The inverse limit is denoted $\varprojlim_\lambda X_\lambda$ and, if it exists, it is unique up to canonical isomorphism compatible with the morphisms giving X and Y the structure of candidates.

We next want to see that inverse limits exist in the categories of sets, abelian groups, rings, R -modules, and R -algebras. The construction for sets also works in the other categories mentioned. Let (Λ, \leq) be a directed partially ordered set and let $(X_\lambda, f_{\lambda,\mu})$ be an inverse limit system of sets. Consider the subset $X \subseteq \prod_\lambda X_\lambda$ consisting of all elements x of the product such that for $\lambda \leq \mu$, $f_{\lambda,\mu}(x_\mu) = x_\lambda$, where x_λ and x_μ are the λ and μ coordinates, respectively, of x . It is straightforward to verify that X is an inverse limit for the system: the maps $X \rightarrow X_\lambda$ are obtained by composing the inclusion of X in the product with the product projections π_λ mapping the product to X_λ .

If each X_λ is in one of the categories specified above, notice that the Cartesian product is as well, and the set X is easily verified to be a subobject in the appropriate category. In every instance, it is straightforward to check that X is an inverse limit.

Suppose, for example, that X_λ is a family of subsets of A ordered by \supseteq , and that the map $X_\mu \rightarrow X_\lambda$ for $X_\lambda \supseteq X_\mu$ is the inclusion of $X_\mu \subseteq X_\lambda$. The condition for the partially ordered set to be directed is that for all λ and μ , there is a set in the family contained in $X_\lambda \cap X_\mu$. The construction for the inverse limit given above yields all functions on these sets with a constant value in the intersection of all of them. This set evidently may be identified with $\bigcap_\lambda X_\lambda$.

We are particularly interested in inverse limit systems indexed by \mathbb{N} . To give such a system one needs to give an infinite sequence of objects X_0, X_1, X_2, \dots in the category and for every $i \geq 0$ a map $X_{i+1} \rightarrow X_i$. The other maps needed can be obtained from these by composition. In the cases of the categories mentioned above, to give an element of the inverse limit is the same as giving a sequence of elements x_0, x_1, x_2, \dots such that for all i , $x_i \in X_i$, and x_{i+1} maps to x_i for all $i \geq 0$. One can attempt to construct an element of the inverse limit by choosing an element $x_0 \in X_0$, then choosing an element $x_1 \in X_1$ that maps to x_0 , etc. If the maps are all surjective, then given $x_i \in X_i$ one can always find an element of the inverse limit that has x_i as its i th coordinate: for $h < i$, use the image of x_i in X_h , while for $i+1, i+2, \dots$ one can choose values recursively, using the surjectivity of the maps.

We want to use these ideas to describe the I -adic completion of a ring R , where R is a ring and $I \subseteq R$ is an ideal. We give two alternative descriptions. Consider the set of all

sequences of elements of R indexed by \mathbb{N} under termwise addition under multiplication: this ring is the same as the product of a family of copies of R index by \mathbb{N} . Let $\mathfrak{C}_I(R)$ denote the subring of *Cauchy sequences for the I -adic topology*: by definition these are the sequences such that for all $t \in \mathbb{N}$ there exists $N \in \mathbb{N}$ such that for all $i, j \geq N$, $r_i - r_j \in I^t$. This is a subring of the ring of sequences. It is an R -algebra via the map $R \rightarrow \mathfrak{C}_I(R)$ that sends $r \in R$ to the constant sequence r, r, r, \dots . Let $\mathfrak{C}_i^0(R)$ be the set of *Cauchy sequences that converge to 0*: by definition, these are the sequences such that for all $t \in \mathbb{N}$ there exists $N \in \mathbb{N}$ such that for all $i \geq N$, $r_i \in I^t$. These sequences are automatically Cauchy. Then $\mathfrak{C}_I^0(R)$ is an ideal of $\mathfrak{C}_I(R)$. It is easy to verify that every subsequence of a Cauchy sequence is again Cauchy, and that it differs from the original sequence by an element of $\mathfrak{C}_I^0(R)$.

Given an element of $\mathfrak{C}_I(R)$, say r_0, r_1, r_2, \dots we may consider the residue mod I^t for a given t . These are eventually all the same, by the definition of a Cauchy sequence. The stable value of these residues is an element of R/I^t , and we thus have a map $\mathfrak{C}_I(R) \rightarrow R/I^t$ that is easily seen to be a ring homomorphism that kills $\mathfrak{C}_I^0(R)$. Therefore, for all t we have a surjection $\mathfrak{C}_I(R)/\mathfrak{C}_I^0(R) \rightarrow R/I^t$. These maps make $\mathfrak{C}_I(R)/\mathfrak{C}_I^0(R)$ a candidate for $\varprojlim_t (R/I^t)$, and so induce a ring homomorphism $\mathfrak{C}_I(R)/\mathfrak{C}_I^0(R) \rightarrow \varprojlim_t R/I^t$.

This map is an isomorphism. Given a sequence of elements in the rings R/I^t that determine an element of the inverse limit, for each residue ρ_t choose an element r_t of R that represents it. It is straightforward to verify that the r_t form a Cauchy sequence in R and that it maps to the element of $\varprojlim_t R/I^t$ with which we started. Consider any other Cauchy sequence with the same image. It is again straightforward to verify that the difference of the two Cauchy sequences is in $\mathfrak{C}_I^0(R)$. This proves the isomorphism:

Theorem. *Let R be any ring and I any ideal. Then $\mathfrak{C}_I(R)/\mathfrak{C}_I^0(R) \rightarrow \varprojlim_t (R/I^t)$ is an isomorphism, and the kernel of the map from R to either of these isomorphic R -algebras is $\bigcap_t I^t$. \square*

These isomorphic rings are denoted \widehat{R}^I or simply \widehat{R} , if I is understood, and either is referred to as the *I -adic completion* of R . If $I \subseteq R$, then R is called *I -adically separated* if $\bigcap_t I^t = (0)$, and *I -adically complete* if $R \rightarrow \widehat{R}^I$ is an isomorphism: this holds iff R is I -adically separated, and every Cauchy sequence is the sum of a constant sequence r, r, r, \dots and a sequence that converges to 0. The Cauchy sequence is said to *converge* to r .

Given a Cauchy sequence in R with respect to I , we may choose a subsequence such that the residues of all terms from the t th on are constant mod I^{t+1} . Call such a Cauchy sequence *standard*. Given a standard Cauchy sequence, let $s_0 = r_0$ and $s_{t+1} = r_{t+1} - r_t \in I^t$ for $t \geq 0$. Then the $s_0 + \dots + s_t = r_t$. Thus, the partial sums of the “formal series” $s_0 + s_1 + s_2 + \dots$ form a Cauchy sequence, and if the ring is complete it converges. Given any formal series $\sum_{t=0}^{\infty} s_t$ such that $s_t \in I^t$ for all t , the partial sums form a Cauchy sequence, and every Cauchy sequence is obtained, up to equivalence (i.e., up to adding a sequence that converges to 0) in this way.

Proposition. *Let J denote the kernel of the map from $\widehat{R}^I \rightarrow R/I$ (J consists of elements represented by Cauchy sequences all of whose terms are in I). Then every element of \widehat{R}^I*

that is the sum of a unit and an element of J is invertible in \widehat{R}^I . Every maximal ideal of \widehat{R}^I contains J , and so there is a bijection between the maximal ideals of \widehat{R}^I and the maximal ideals of R/I . In particular, if R/I is quasi-local, then \widehat{R}^I is quasi-local.

Proof. If u is a unit and $j \in J$ we may write $u = u(1 + u^{-1}j)$, and so it suffices to show that $1 + j$ is invertible for $j \in J$. Let r_0, r_1, \dots be a Cauchy sequence that represents j . Consider the sequence $1 - r_0, 1 - r_1 + r_1^2, \dots, 1 - r_n + r_n^2 - \dots + (-1)^{n-1}r_n^{n+1}, \dots$: call the n th term of this sequence v_n . If r_n and r_{n+1} differ by an element of I^t , then v_n and v_{n+1} differ by an element of $I^t + I^{n+2}$. From this it follows that v_n is a Cauchy sequence, and $1 - (1 + r_n)v_n = r_n^{n+2}$ converges to 0. Thus, the sequence v_n represents an inverse for $1 + j$ in \widehat{R}^I .

Suppose that m is a maximal ideal of \widehat{R}^I and does not contain $j \in J$. Then j has an inverse $v \bmod m$, so that we have $ju = 1 + u$ where $u \in m$, and then $-u = 1 - ju$ is not invertible, a contradiction, since $ju \in J$. \square

Suppose that $\bigcap_t I^t = 0$. We define the distance $d(r, s)$ between two elements $r, s \in R$ to be 0 if $r = s$, and otherwise to be $1/2^n$ (this choice is somewhat arbitrary), where n is the largest integer such that $r - s \in I^n$. This is a metric on R : given three elements $r, s, t \in R$, the triangle inequality is clearly satisfied if any two of them are equal. If not, let n, p, q be the largest powers of I containing $r - s$, $s - t$, and $t - r$, respectively. Since $t - r = -(s - t) - (r - s)$, $q \geq \min\{n, p\}$, with equality unless $n = p$. It follows that in every “triangle,” the two largest sides (or all three sides) are equal, which implies the triangle inequality. The notion of Cauchy sequence that we have given is the same as the notion of Cauchy sequence for this metric. Thus, \widehat{R}^I is literally the completion of R as a metric space with respect to this metric.

Given a ring homomorphism $R \rightarrow R'$ mapping I into an ideal I' of R' , Cauchy sequences in R with respect to I map to Cauchy sequences in R' with respect to I' , and Cauchy sequences that converge to 0 map to Cauchy sequences that converge to 0. Thus, we get an induced ring homomorphism $\widehat{R}^I \rightarrow \widehat{R}'^{I'}$. This construction is functorial in the sense that if we have a map to a third ring R'' , a ring homomorphism $R' \rightarrow R''$, and an ideal I'' of R'' such that I' maps into I'' , then the induced map $\widehat{R}^I \rightarrow \widehat{R}''^{I''}$ is the composition $(\widehat{R}'^{I'} \rightarrow \widehat{R}''^{I''}) \circ (\widehat{R}^I \rightarrow \widehat{R}'^{I'})$. If $R \rightarrow R'$ is surjective and I maps onto I' , then the map of completions is surjective: each element of $\widehat{R}'^{I'}$ can be represented as the partial sums of a series $s_0 + s_1 + s_2 + \dots$, where $s_n \in (I')^n$. But I^n will map onto $(I')^n$, and so we can find $r_n \in I^n$ that maps to s_n , and then $r_0 + r_1 + r_2 + \dots$ represents an element of \widehat{R}^I that maps to $s_0 + s_1 + s_2 + \dots$.

Example. Let $S = R[x_1, \dots, x_n]$ be the polynomial ring in n variables over R , and let $I = (x_1, \dots, x_n)S$. An element of S/I^n is represented by a polynomial of degree $\leq n - 1$ in the x_i . A sequence of such polynomials will represent an element of the inverse limit if and only if, for every n , then n th term is precisely the sum of the terms of degree at most n in the $n + 1$ st term. It follows that the inverse limit ring \widehat{S}^I is $R[[x_1, \dots, x_n]]$, the formal power series ring. In consequence, we can prove:

Theorem. *If R is a Noetherian ring and I is an ideal of R , then \widehat{R}^I is Noetherian.*

Proof. Suppose that $I = (f_1, \dots, f_n)R$. Map the polynomial ring $S = R[x_1, \dots, x_n]$ to R as an R -algebra by letting $x_j \mapsto f_j$. This is surjective, and $(x_1, \dots, x_n)S$ maps onto I . Therefore we get a surjection $R[[x_1, \dots, x_n]] \rightarrow \widehat{R}^I$. Since we already know that the formal power series ring is Noetherian, it follows that \widehat{R}^I is Noetherian. \square

We next want to form the I -adic completion of an R -module M . This will be not only an R -module: it will also be a module over \widehat{R}^I . Let R be a ring, $I \subseteq R$ an ideal and M an R -module. Let $\mathfrak{C}_I(M)$ denote the Cauchy sequences in M with respect to I : the sequence u_0, u_1, u_2, \dots is a *Cauchy sequence* if for all $t \in \mathbb{N}$ there exists $N \in \mathbb{N}$ such that $u_i - u_j \in I^t M$ for all $i, j \geq N$. These form a module over $\mathfrak{C}_I(R)$ under termwise multiplication, and set of Cauchy sequences, $\mathfrak{C}_I^0(M)$, that converge to 0, where this means that for all t , the terms of the sequence are eventually all in $I^t M$, is a submodule that contains $\mathfrak{C}_I^0(R)\mathfrak{C}_I(M)$. The quotient $\mathfrak{C}_I(M)/\mathfrak{C}_I^0(M)$ is consequently a module over \widehat{R}^I . Moreover, any homomorphism $h : M \rightarrow N$ induces a homomorphism from $\mathfrak{C}_I(M) \rightarrow \mathfrak{C}_I(N)$ that preserves convergence to 0, and hence a homomorphism $\widehat{h}^I : \widehat{M}^I \rightarrow \widehat{N}^I$. This is a covariant functor from R -modules to \widehat{R}^I -modules. There is an R -linear map $M \rightarrow \widehat{M}^I$ that sends the element u to the element represented by the constant Cauchy sequence whose terms are all u . The kernel of this map is $\bigcap_t I^t M$, and so it is injective if and only if $\bigcap_t I^t M = 0$, in which case M is called *I -adically separated*. If $M \rightarrow \widehat{M}^I$ is an isomorphism, M is called *I -adically complete*. The maps $M \rightarrow \widehat{M}^I$ give a natural transformation from the identity functor on R -modules to the I -adic completion functor. Moreover, by exactly the same reasoning as in the case where $M = R$, $\widehat{M}^I \cong \varprojlim_t M/I^t M$.

I -adic completion commutes in an obvious way with finite direct sums and products (which may be identified in the category of R -modules). The point is that $u_n \oplus v_n$ gives a Cauchy sequence (respectively, a sequence converging to 0) in $M \oplus N$ if and only if u_n and v_n give such sequences in M and N . Moreover if $f_1 : M_1 \rightarrow N$ and $f_2 : M_2 \rightarrow N$, we have that the I -adic completion of the map $f_1 \oplus f_2 : M_1 \oplus M_2 \rightarrow N$ is the direct sum of the completions, $\widehat{f}_1 \oplus \widehat{f}_2$. A similar remark applies when we have $g_1 : M \rightarrow N_1$ and $g_2 : M \rightarrow N_2$, and we consider the map $(g_1, g_2) : M \rightarrow N_1 \times N_2$. The situation is the same for finite direct sums and finite direct products. Note also that if we consider the map given by multiplication by r on M , the induced endomorphism of \widehat{M}^I is given by multiplication by r (or by the image of r in \widehat{R}^I).

If $M \rightarrow Q$ is surjective, the map $\widehat{M}^I \rightarrow \widehat{Q}^I$ is surjective: as in the case of rings, any element z of \widehat{Q}^I can be represented using the Cauchy sequence of partial sums of a formal series $q_0 + q_1 + q_2 + \dots$ where $q_t \in I^t Q$. To see this, take a Cauchy sequence that represents the element. Pass to a subsequence w_0, w_1, w_2, \dots such that the residue of w_k in $M/I^t M$ is the same for all $k \geq t$. The element can be thought of as

$$w_0 + (w_1 - w_0) + (w_2 - w_1) + \dots .$$

Thus, take $q_0 = w_0$ and $q_t = w_t - w_{t-1}$ for $t \geq 1$. For all t , $I^t M$ maps onto $I^t Q$. Therefore we can find $u_t \in I^t M$ such that u_t maps to q_t , and the partial sums of $u_0 + u_1 + u_2 + \dots$ represent an element of \widehat{M}^I that maps to z .

Note that because \widehat{M}^I is an R -module and we have a canonical map $M \rightarrow \widehat{M}^I$ that is R -linear, the universal property of base change determines a map $\widehat{R}^I \otimes_R M \rightarrow \widehat{M}^I$. These maps give a natural transformation from the functor $\widehat{R}^I \otimes_R _$ to the I -adic completion functor: these are both functors from R -modules to \widehat{R}^I -modules. If M is finitely generated over a Noetherian ring R , this map is an isomorphism: not only that: restricted to finitely generated modules, I -adic completion is an exact functor, and \widehat{R}^I is flat over R .

In order to prove this, we need to prove the famous Artin-Rees Lemma. Let R be a ring and I an ideal of R . Let t be an indeterminate, and let $It = \{it : i \in I\} \subseteq R[t]$. Then $R[It] = R + It + I^2t^2 + \dots$ is called the *Rees ring* of I . If $I = (f_1, \dots, f_n)$ is finitely generated as an ideal, then $R[It] = R[f_1t, \dots, f_nt]$ is a finitely generated R -algebra. Therefore, the Rees ring is Noetherian if R is.

Before proving the Artin-Rees theorem, we note that if M is an R -module and t and indeterminate, then every element of $R[t] \otimes M$ can be written uniquely in the form

$$1 \otimes u_0 + t \otimes u_1 + \dots + t^k \otimes u_k,$$

where the $u_j \in M$, for any sufficiently large k : if a larger integer s is used, then one has $m_{k+1} = \dots = m_s = 0$. This is a consequence of the fact that $R[t]$ is R -free with the powers of t as a free basis. Frequently one writes $u_0 + u_1t + \dots + u_kt^k$ instead, which looks like a polynomial in t with coefficients in M . When this notation is used, $M[t]$ is used as a notation for the module. Note that the $R[t]$ -module structure is suggested by the notation: $(rt^j)(ut^k) = (ru)t^{j+k}$, and all other more general instances of multiplication are then determined by the distributive law.

We are now ready to prove the Artin-Rees Theorem, which is due independently to Emil Artin and David Rees.

Theorem (E. Artin, D. Rees). *Let $N \subseteq M$ be Noetherian modules over the Noetherian ring R and let I be an ideal of R . Then there is a constant positive integer c such that for all $n \geq c$, $I^n M \cap N = I^{n-c}(I^c M \cap N)$. That is, eventually, each of the modules $N_{n+1} = I^{n+1}M \cap N$ is I times its predecessor, $N_n = I^n M \cap N$.*

In particular, there is a constant c such that $I^n M \cap N \subseteq I^{n-c}N$ for all $n \geq c$. In consequence, if a sequence of elements in N is an I -adic Cauchy sequence in M (respectively, converges to 0 in M) then it is an I -adic Cauchy sequence in N (respectively, converges to 0 in N).

Proof. We consider the module $R[t] \otimes M$, which we think of as $M[t]$. Within this module,

$$\mathcal{M} = M + IMt + I^2Mt^2 + \dots + I^kMt^k + \dots$$

is a finitely generated $R[It]$ -module, generated by generators for M as an R -module: this is straightforward. Therefore, \mathcal{M} is Noetherian over $R[It]$. But

$$\mathcal{N} = N + (IM \cap N)t + (I^2M \cap N)t^2 + \dots,$$

which may also be described as $N[t] \cap \mathcal{M}$, is an $R[It]$ submodule of \mathcal{M} , and so finitely generated over $R[It]$. Therefore for some $c \in \mathbb{N}$ we can choose a finite set of generators whose degrees in t are all at most c . By breaking the generators into summands homogeneous with respect to t , we see that we may use elements from

$$N, (IM \cap N)t, (I^2M \cap N)t^2, \dots, (I^cM \cap N)t^c$$

as generators. Now suppose that $n \geq c$ and that $u \in I^nM \cap N$. Then ut^n can be written as an $R[It]$ -linear combination of elements from

$$N, (IM \cap N)t, (I^2M \cap N)t^2, \dots, (I^cM \cap N)t^c,$$

and hence as a sum of terms of the form

$$i_h t^h v_j t^j = (i_h v_j) t^{h+j}$$

where $j \leq c$, $i_h \in I^h$, and

$$v_j \in I^jM \cap N.$$

Of course, one only needs to use those terms such that $h + j = n$. This shows that $(I^nM) \cap N$ is the sum of the modules

$$I^{n-j}(I^jM \cap N)$$

for $j \leq c$. But

$$I^{n-j}(I^jM \cap N) = I^{n-c}I^{c-j}(I^jM \cap N),$$

and

$$I^{c-j}(I^jM \cap N) \subseteq I^cM \cap N,$$

so that we only need the single term $I^{n-c}(I^cM \cap N)$. \square

Theorem. *Let R be a Noetherian ring, $I \subseteq R$ an ideal.*

- (a) *If $0 \rightarrow N \rightarrow M \rightarrow Q \rightarrow 0$ is a short exact sequence of finitely generated R -modules, then the sequence $0 \rightarrow \widehat{N}^I \rightarrow \widehat{M}^I \rightarrow \widehat{Q}^I \rightarrow 0$ is exact. That is, I -adic completion is an exact functor on finitely generated R -modules.*
- (b) *The natural transformation θ from $\widehat{R}^I \otimes_R _$ to the I -adic completion functor is an isomorphism of functors on finitely generated R -modules. That is, for every finitely generated R -module M , the natural map $\theta_M : \widehat{R}^I \otimes_R M \rightarrow \widehat{M}^I$ is an isomorphism.*
- (c) *\widehat{R}^I is a flat R -algebra. If (R, \mathfrak{m}) is local, $\widehat{R} = \widehat{R}^{\mathfrak{m}}$ is a faithfully flat local R -algebra.*

Proof. (a) We have already seen that the map $\widehat{M}^I \rightarrow \widehat{Q}^I$ is surjective. Let y be an element of \widehat{M}^I that maps to 0 in \widehat{Q}^I . Choose a Cauchy sequence that represents z , say u_0, u_1, u_2, \dots . After passing to a subsequence we may assume that $u_t - u_{t+1} \in I^tM$ for every t . The images of the u_t in $Q \cong M/N$ converge to 0. Passing to a further subsequence we may assume that the image of $u_t \in I^t(M/N)$ for all t , so that $u^t \in I^tM + N$, say $u_t = v_t + w_t$ where $v_t \in I^tM$ and $w_t \in N$. Then w_t is a Cauchy sequence in M that represents z : in

fact, $w_t - w_{t+1} \in I^t M \cap N$ for all t . Each $w_t \in N$, and so the elements w_t form a Cauchy sequence in N , by the Artin-Rees Theorem. Thus, every element in $\text{Ker}(\widehat{M}^I \rightarrow \widehat{Q}^I)$ is in the image of \widehat{N}^I .

Finally, suppose that z_0, z_1, z_2, \dots is a Cauchy sequence in N that converges to 0 in M . Then z_t already converges to 0 in N , and this shows that \widehat{N}^I injects into \widehat{M}^I . This completes the proof of part (a).

(b) Take a presentation of M , say $R^n \xrightarrow{A} R^m \rightarrow M \rightarrow 0$, where $A = (r_{ij})$ is an $m \times n$ matrix over R . This yields a diagram:

$$\begin{array}{ccccccc} \widehat{R}^I \otimes_R R^n & \xrightarrow{A} & \widehat{R}^I \otimes_R R^m & \longrightarrow & \widehat{R}^I \otimes_R M & \longrightarrow & 0 \\ \theta_{R^n} \uparrow & & \theta_{R^m} \uparrow & & \theta_M \uparrow & & \\ \widehat{R}^n & \xrightarrow{A} & \widehat{R}^m & \longrightarrow & \widehat{M} & \longrightarrow & 0 \end{array}$$

where the top row is obtained by applying $\widehat{R}^I \otimes _$, and is exact by the right exactness of tensor, the bottom row is obtained by applying the I -adic completion functor, and is exact by part (a). The vertical arrows are given by the natural transformation θ , and the squares commute because θ is natural. The map θ_{R^h} is an isomorphism for $h = m$ or $h = n$ because both functors commute with direct sum, and the case where the free module is just R is obvious. But then θ_M is an isomorphism, because cokernels of isomorphic maps are isomorphic.

(c) We must show that $\widehat{R}^I \otimes_R N \rightarrow \widehat{R}^I \otimes_R M$ is injective for every pair of R -modules $N \subseteq M$. We know this from parts (a) and (b) when the modules are finitely generated. The result now follows from the Lemma just below. Faithful flatness is clear, since the maximal ideal of R clearly expands to a proper ideal in \widehat{R}^I . \square

Lemma. *Let F be an R -module, and suppose that whenever $N \subseteq M$ are finitely generated R -modules then $F \otimes_R N \rightarrow F \otimes_R M$ is injective. Then F is flat.*

Proof. Let $N \subseteq M$ be arbitrary R -modules. Then $F \otimes_R N$ is the directed union of the images of the modules $F \otimes_R N_0$ as F runs through the finitely generated submodules of M . Thus, if $z \in F \otimes_R N$ maps to 0 in $F \otimes_R M$, it will be the image of $z' \in N_0 \otimes M - \{0\}$, which implies that $z' \in F \otimes_R N_0$ maps to 0 in $F \otimes_R M$. But since M is the directed union of its finitely generated modules M_0 containing N_0 , and since $F \otimes_R M$ is the direct limit of these, it follows that for some sufficiently large but finitely generated $M_0 \supseteq N_0$, the image of z' under the map $F \otimes_R N_0 \rightarrow F \otimes_R M_0$ is 0. But then $z' = 0$ and so $z = 0$, as required. \square