Math 615, Winter 2011 **Problem Set #1**
Due: January 27.

1. Show carefully that if I is finitely generated, and $S = \widehat{R}^I$, then I^nS may be identified with the images of Cauchy sequences over R all of whose terms are eventually in I^n. Conclude that $S/I^nS \cong R/I^n$.

2. Let I and J be ideals of R such that R is complete with respect to I and is also complete with respect to J. Show that for every Cauchy sequence $r_0, r_1, r_2, \ldots r_n, \ldots$ with respect to $I + J$, there is an element $r \in R$ such that for every positive integer k, the sequence $r - r_1, r - r_2, \ldots, r - r_n, \ldots$ has terms that are eventually in $(I + J)^k$. Hence, if R is $(I + J)$-adically separated, then $R \cong \widehat{R}^{(I+J)}$.

3. Let $x_1, \ldots, x_n, \ldots$ be a countably infinite sequence of indeterminates over a field K, let $A = K[x_1, \ldots, x_n, \ldots]$ be a polynomial ring and let I be the ideal of A generated by all the x_i. Let R denote the I-adic completion of A. Show that $\sum_{t=n}^{\infty} x_t^t$ is not in I^nR.

4. Let $R = K[x_1, \ldots, x_m, y_1, \ldots, y_n]$. Let S denote the completion of R with respect to the ideal $I = (x_1, \ldots, x_m)R$, which you may assume is the formal power series ring in the x_i with coefficients in $K[y_1, \ldots, y_n]$. Note that S injects into $K[[x_1, \ldots, x_m, y_1, \ldots, y_n]]$, which may be identified with $A[[y_1, \ldots, y_n]]$, where $A = K[[x_1, \ldots, x_m]]$. Let $P = (x_1, \ldots, x_m)A$, Prove that the image of S is the subring of $A[[y_1, \ldots, y_n]]$ consisting of those formal powers series f in $y_1, \ldots, y_n$ such that for every positive integer k, all but finitely many of the coefficients of f lie in P^k.

5. Let (R, m, K) be a complete local ring whose residue class field K contains all of the n th roots of 1. Suppose that K has characteristic 0 or that the characteristic of K does not divide n. Prove that for every element $u \in m$, the element $1 + u$ has n distinct n th roots in R. In particular, conclude that if the characteristic of K is not 2, $1 + u$ has two distinct square roots, while if $K = \mathbb{C}$, every unit has n distinct n th roots.

6. Show that the formal power series ring $\mathbb{C}[[x]]$ has a coefficient field K that contains $\pi + x$, where $\pi \in \mathbb{C}$ has its usual meaning (i.e., it is the transcendental number $3.14159\ldots$).

EXTRA CREDIT 1. In **#2.** above, is it necessary to assume that R is $(I + J)$-adically separarted, or does this follow from completeness with respect to both I and J?

EXTRA CREDIT 2. Let K be a field and X an indeterminate. Let f be an irreducible monic polynomial of degree at least 2 in $K[X]$ and let $P = fK[X]$. Let $V = K[X]_P$.

(a) Show that if $K = \mathbb{R}$ and $f = X^2 + 1$, V does not have a coefficient field.

(b) Show, in general, that there is no coefficient field for V that contains K.

(c) Is there any choice of K and f such that V has a coefficient field if one does not require it to contain the "original" copy of K?