Math 615, Winter 2012 **Problem Set #1: Solutions**

1. Since I^n is also finitely generated the powers of I and I^n are cofinal, we may replace I by I^n. It therefore suffices to consider the case $n = 1$. Let $I = (f_1, \dots, f_h)$. Clearly, J be the ideal of S which is the image of the Cauchy sequences with all entries in I. It is clear that $IS \subseteq J$. It therefore suffices to show that if $\{r_n\}_n$ is a Cauchy sequence with entries in I, there is an equivalent sequence in IS. As in class, we may replace the given sequence with a subsequence with the property that for $j \geq i$, $r_j - r_i \in I^i$. If $s_1 = r_1$ and $s_n = r_n - r_{n-1}$ for $n \geq 1$ the given sequence is the partial sums of $\sum_{n=0}^{\infty} s_n$, where $s_n \in I^n = II^{n-1} = (f_1, \dots, f_h)I^{n-1}$, so that for each n we may write $s_n = \sum_{j=1}^{h} f_j s_{n,j}$ with $s_{n,j} \in I^{n-1}$ for all $n \geq 1$ and all j. Let $\rho_j = \{r_{n,j}\}_n$ be the sequence of partial sums of $\sum_{n=1}^{\infty} s_{n,j}$. This is clearly Cauchy, and $\{r_n\}_n = \sum_{j=1}^{h} f_j \rho_j$, as required. $\square$
Given an element of the completion, it is represented by a Cauchy sequence. The residues of the terms mod I are eventually all the same. This gives an obvious surjection $S \twoheadrightarrow R/I$. The kernel evidently consists of Cauchy sequences whose terms are eventually in I. Such a sequence is equivalent to a subsequence whose terms are all in I, by omittting finitely many terms, and hence to a sequence representing an element of IS by the first part. $\square$

2. Since a monomial with $2n - 1$ factors, each from I or J, must have at least n factors from I or n from J, we have that $(I + J)^{2n-1} \subseteq I^n + J^n \subseteq (I + J)^n$. Hence, a Cauchy sequence with respect to $I + J$ has a subsequence $\{r_n\}_n$ such that $r_t - r_n \in I^n + J^n$ for all $t \geq n$. This sequence can be thought of as partial sums of a series $\sum_{s=0}^{\infty} s_n$ where $s_n \in I^n + J^n$. Let $s_n = i_n + j_n$ where $i_n \in I^n$ and $j_n \in J^n$. Then the partial sums of $\sum_{n=0}^{\infty} i_n$ (resp., $\sum_{n=0}^{\infty} j_n$) are Cauchy with respect to I (resp., J) and have a limit u (resp., v) in S, by completeness with respect to I (resp., J). It follows that $u + v$ has the required property. $\square$

3. The sum is not even in I. If it were, it would be a linear combination of finitely many of the generators x_i, and hence of $x_1, \dots, x_k$ for some sufficiently large k. But every element of the ring has a unique expression as a formal power series $\sum_{i=0}^{\infty} F_i$ where the F_i are polynomial forms that are either 0 or degree i, and any element in the ideal generated by $x_1, \dots, x_k$ has the property that all of the terms that occur in the power series expansion consist of monomials that are divisible by at least one of $x_1, \dots, x_k$. The given element has the property that it has a term x_N^N for $N > k$ which is not in $(x_1, \dots, x_k)R$. $\square$

4. Every $f \in K[[x_1, \dots, x_m, y_1, \dots, y_n]]$ is an infinite linear combination over K of monomials in all the variables each of which can written uniquely in the form $\mu\nu$, whereμ (resp., ν) is a monomial in $x_1, \dots, x_m$ (resp., $y_1, \dots, y_n$). The condition that f be in the image is that for each monomials μ in $x_1, \dots, x_m$, the number of terms of the form $\mu\nu_i$ for some monomial ν_i in $y_1, \dots, y_n$ is finite, for this means precisely that the coefficient of μ in $K[[y_1, \dots, y_n]]$ is actually in $K[y_1, \dots, y_n]$. This is false for a given μ, then μ occurs as a coefficient for infinitely many monomials ν in $y_1, \dots, y_n$, and since we can choose k so that $\mu \notin P^k$, this contradicts the given condition. On the other hand, if the condition fails, there exists k and infinitely many ν_i with a coefficient not in P^k. Since there are only finitely many monomials in $x_1, \dots, x_m$ not in P^k and a least one of these must occur as a coefficient in one of the ν_i, by the pigeonhole principle one of them, call it μ must occur in a term $c\mu\nu_i$ of f (where $c \in K - \{0\}$) for infinitely many choice of i. $\square$

5. By Hensel's lemma, $z^n - (1+u) = 0$ will factor into n linear factors, lifting the factorization of $z^n - 1 = 0$ mod m, which will give n roots corresponding to the n distinct nth roots of unity in the residue class field. The distinctness of the roots is needed so that the linear factors over K will be relatively prime in pairs. The final assertions are immediate.

6. Let $A = \mathbb{Q}[\pi + x] \in \mathbb{C}[[x]]$. The image of any nonzero element $f(\pi + x)$ of A, where f is ,a nonzero polynomial in one variable over $\mathbb{Q}$ mod $x\mathbb{C}[[x]]$ is $f(\pi)$, which is nonzero because π is transcendental over $\mathbb{Q}$. Hence, every element of the multiplicative system $W = A - \{0\}$ is a unit in $\mathbb{C}[[x]]$, and has an inverse in that ring. It follows from the universal mapping property of localization that we have a nonzero map $W^{-1}A \to \mathbb{C}[[x]]$, which is an injection since $W^{-1}A$ is the field $\mathbb{Q}(\pi + x)$. Thus, $\mathbb{Q}(\pi + x)$ is a field contained in $\mathbb{C}[[x]]$. By a class theorem, every subfield of characteristic 0 in a complete local ring can be enlarged to a coefficient field. $\square$

EXTRA CREDIT 1. We construct R with ideals I, $J \subseteq R$ such that R is complete and separated with respect to I and J but not separated with respect to $I + J$. Adjoin countably many indeterminates $\underline{x} = x_{ij}, \underline{y} = y_{ij}$ $1 \leq j \leq i < \infty$, to a field K. Kill all x_{ij}^n, $n > 1$, all $x_{ij}y_{i'j'}$, and all $x_{ij}x_{i'j'}, y_{ij}y_{i'j'}$ for $i \neq i'$. Let $u_n = \prod_{j=1}^n x_{nj}$ and $v_n = \prod_{j=1}^n y_{nj}$. u_n and v_n both kill $(\underline{x}, \underline{y})$ for all n. Complete with respect to the ideal $(\underline{x}, \underline{y})$ generated by all $x_{ij}, y_{i'j'}$. In the resulting ring T, every element is uniquely a formal sum $\sum_{n=0}^{\infty} F_n$ where each F_n is the sum of polynomial in the x_{ij} of total degree n whose degree in each x_{ij} is 1 (and such that if x_{ij}, $x_{i'j'}$ occur in a term, then $i = i'$) and a polynomial in the y_{ij} satisfying corresponding conditions. There is a maximal ideal $\mathcal{M}$ consisting of formal sums in which the constant is 0, and it is killed by both u_n and v_n. Hence, $\mathcal{M}$ also killed by all the $g_n = u_n + v_n \in (I + J)^n$. Let $\mathfrak{A}$ be the ideal generated by the $g_n - g_{n+1}$. Then any element of $\mathfrak{A}$ is a finite K-linear combination of the $g_n - g_{n+1}$ and, hence, a finite K-linear combination of the g_n in which the sum of the coefficients is 0. Let $R = T/\mathfrak{A}$. We show that R is complete and separated with respect to both $I = (\underline{x})R$ and $J = (\underline{y})R$, but that $g_1 \in \bigcap_n (I + J)^n$ and $g_1 \neq 0$.

To see that g_1 is not 0, it suffices to show g_1 is not a finite K-linear combination of the $g_n - g_n + 1$. This is clear, since the g_i are linearly independent over K: in fact, even the $\underline{y}$ terms of the g_i, which are the v_i, are linearly independent over K.

By symmetry, it suffices to prove I-adic separation (every I-adic Cauchy sequence has a limit, since this is true in T). An element in $(\underline{x})^n T$ is in $(x_{ij} : i \leq s)^n$ for some s. Hence, in its unique representation there are no terms involving the $\underline{y}$, nor any x_{mj} for $j > s$. Hence, the element is represented by a polynomial H_n in the $\underline{x}$ all of whose terms have degree at least n. Thus, a nonzero element of $R = T/\mathfrak{A}$ in $\bigcap_n I^n$ has a sequence of nonzero polynomial representatives $H_n \in T$ such that the difference of any two, $H_N - H_n$, is a finite K-linear combination of the g_t. But this K-linear combination must be 0, since $H_N - h_n$ has no terms in the $\underline{y}$ and the v_t are linearly independent over K. Hence H_N and $H_n \neq 0$ cannot be equal in R when N is larger than the degree of H_n unless both are 0. $\square$

EXTRA CREDIT 2. (a) No, since -1 has no square root in even in the fraction field of the ring, which $\mathbb{R}(x)$.

(b) Likewise, $K(x)$ does not contain an element algebraic over K but not in K.

(c) Yes, this can happen. Let k be any field and let t, x be indeterminates over K. Let $K = k(t)$, and let $f = x^2 - t$, which is irreducibe. In this case, $K[x]_P$ has a coefficient field! First note that $k[x] \subseteq K[x]_P$ meets the maximal in 0: it maps to the quotient $V/PV \cong K(\sqrt{t}) = k(\sqrt{t})$ so that x maps to $\sqrt{t}$ while k is fixed. Hence, $K(x) \subseteq R_P$, and is mapped isomorphically to the residue class field $k(\sqrt{t})$ mod PR_P.