Math 615, Winter 2012 **Problem Set #2**
Due: February 20.

1. Let K be a field of characteristic $p > 0$ and Λ a p-base for K. Let $t_1, \ldots, t_n$ be indeterminates over K and let L be the field $K(t_1, \ldots, t_n)$. Show that the set $\Lambda' = \Lambda \cup \{t_1, \ldots, t_n\}$ extends Λ to a p-base for L.

2. Let R and S be finitely generated $\mathbb{N}$-graded algebras over a field K such that $R_0 = S_0 = K$. Let $T = R \otimes_K S$ graded so that $T_n = \bigoplus_{i+j=n} R_i \otimes_K S_j$. Show that $P_T(t) = P_R(t)P_S(t)$. (This gives another proof that the Poincaré series of $K[x_1, \ldots, x_n]$ is $1/(1-t)^n$.)

3. Let $R = K[x_1, x_2]$ be the polynomial ring in two variables over a field K N-graded so that $x_1^i x_2^j$ has degree $i + 2j$. Thus, $\deg(x_i) = i$. What is $P_R(t)$ as a rational function of t? Is $\mathrm{Hilb}_R(n) = \dim_K R_n$ eventually a polynomial in n? Prove your answer.

4. Let R be a finitely generated $\mathbb{N}$-graded algebra over a field K such that $R_0 = K$ and let $F_1, \ldots, F_d$ be a homogeneous system of paramters for R such that $\deg(F_i) = k_i$, $1 \le i \le d$. Assume that F_1 is not a zerodivisor in R, and that the image of F_{i+1} is not zerodivisor in $R/(F_1, \ldots, F_i)$, $1 \le i \le d-1$. Show that $P_R(t)$ can be written as $N(t)/D^(t)$ where $D = \prod_{i=1}^{d}(1 - t^{k_i})$ and $N(t) \in \mathbb{Z}[t]$ is a polynomial in t all of whose nonzero coefficients are positive.

5. Let notation be as in Problem **#2.**. One may give T an $\mathbb{N} \times \mathbb{N}$ grading such that $T_{i,j} = R_i \otimes_K S_j$. Let $R \circledS_K S = \bigoplus_n T_{n,n}$. This ring is called the *Segre product of* R *and* S (the notation $\circledS$ used is not standard). It is always a finitely generated K-algebra, and we give it an $\mathbb{N}$-grading so that $(R \circledS_K S)_n = T_{n,n}$. It is easy to verify that if R and S are standard with $R_0 = S_0 = K$, so is $R \circledS_K S$.

(a) Let $R = K[x_1, \ldots, x_r]$ and $S = K[y_1, \ldots, y_s]$ be polynomial rings. Determine the Hilbert function of $D = R \circledS_K S$ and deduce the Krull dimension of D from the result. Also determine the Poincaré series of D as a rational function.

(b) Let notation be as in (a) with $r = 3$ and $s = 2$. Let $\overline{R} = R/(x_1^3 + x_2^3 + x_3^3)$. What is the Poincaré series of $\overline{R} \circledS_K S$ as a rational function?

6. Let R be a finitely generated $\mathbb{N}$-graded algebra. Let $R^{(d)} = \bigoplus_{n=0}^{\infty} R_{nd}$ $\mathbb{N}$-graded so that $[R^{(d)}]_n = R_{nd}$. ($R^{(d)}$ is called the dth *Veronese subring* of R.) If $R_0 = A$ is Artin local, define $\mathfrak{a}(R) \in \mathbb{Z}$ to be the degree of the Poincaré series $P_R(t)$ of R as a rational function. Let s, $d > 0$ be integers and let $S = R^{(d)}$, where $R = Kx_1, \ldots, x_s]$ is the polynomial ring in s variables over K with the standard grading. Determine $\mathrm{Hilb}_S(n)$, $P_S(t)$, and $\mathfrak{a}(S)$.

EXTRA CREDIT 3. Is there a finitely generated $\mathbb{N}$-graded K-algebra R over a field K such that such that $R_0 = K$ and, for all $n \in \mathbb{N}$, $\dim_K(R_n) = f_{n+1}$, where f_n is the nth Fibonacci number? (That is, $f_0 = 0, f_1 = 1$, and $f_{n+1} = f_n + f_{n-1}$ for $n \ge 1$.) Exhibit such an R, or prove that it cannot exist.

EXTRA CREDIT 4. Let R be a finitely generated $\mathbb{N}$-graded algebra. Show that there exists $d > 0$ such that $R^{(d)}$ is generated over R_0 by $[R^{(d)}]_1 = R_d$.