Math 615, Winter 2012 **Problem Set #2: Solutions**

1. By induction on n, it suffices to consider the case where $n = 1$ and $t = t_1$. Since $K(t^p)$ consists of rational functions such that every exponent occurring in either the numerator or denominator is divisible by p, it is clear that $t \notin K(t^p)$, so that t is p-independent of (every finite subset of Λ). On the other hands, since $K = K^p[\Lambda]$, $K(t) = K^p(\Lambda \cup \{t\}$, and so $\Lambda \cup \{t\}$ is a p=base.

2. $\dim_K[(R \otimes_K S)]_n = \sum_{i+j=n} \dim_K(R_i \otimes_K S_j) = \sum_{i+j=n} \dim_K R_i \dim_K S_j$, since the dimension of the tensor product of two vector spaces over K is the product of their dimensions. This is the same as the coefficient of t^n in the product of the power series $\sum_i (\dim_K R_i)t^i$ and $\sum_j (\dim_K R_j)t^j$.

3. The n th coefficient in the Poincaré series is the number of choices of nonnegative integers i, j such that $i + 2j = n$. Clearly, i is determined by j, and the choices for j are $0, 1, \ldots, \lfloor \frac{n}{2} \rfloor$, where $\lfloor r \rfloor$ is the integer part of r. Hence, the Hilbert function is $\lfloor \frac{n}{2} \rfloor + 1$, which is not polynomial, while the Poincaré series is $1/(1-t)(1-t^2)$, by the argument in the solution of Problem 4 below.

4. By a class result, if F is a nonzerodivisor of degree k, $P_R(t) = \frac{1}{1-t^k} P_{R/(F)}(t)$. By induction on d we obtain that $P_R(t) = \prod_{i=1}^d \frac{1}{1-t^{k_i}} P_{R_0}(t)$, where R_0 is the Artin graded ring $R/(F_1, \ldots, F_d)R$. $P_{R_0}(t)$ is a polynomial with nonnegative coefficients, since the coefficients are the dimensions of the graded pieces of R_0. There are only finitely many nonzero coefficients because R_0 is Artin. (In Problem 3 we may use x_1, x_2 as the system of paramters, and $R_0 = K$.) $\square$

5. Suppose that R, S are finitely generatd $\mathbb{N}$-graded K-algebras with $R_0 = S + 0 = K$. Let $f(t) = P_R(t) = a_0 + a_1 t + \cdots + a_n t^n + \cdots$ and $P_S(t) = b_0 + b_1 t + \cdots b_n t^n + \cdots$. $P_T(t) = \sum_{n=0}^{\infty} a_n b_n t^n$. The Hilbert function is the product of the Hilbert functions. If R, S are generated in degree one, its degree is therefore $\dim(R) - 1 + \dim(S) - 1$, which must be $\dim(T) - 1$. Hence, $\dim(T) = \dim(R) + \dim(S) - 1$. Now let R be as above, but arbitrary, and $S = K[x_1, \ldots, x_s]$. $P_T(t) = \sum_{n=0}^{\infty} a_n \binom{s+n-1}{s-1} t^n$. Then $P_T(t) = \frac{1}{(s-1)!} D^{s-1}\big(f(t)t^{s-1}\big)$ where D^k is the k th derivative with respect to t. If $s = 2$, this is simply $f(t) + tf'(t)$.

Since $D^k(fg) = \sum_{j=0}^k \binom{k}{j} D^j(f) D^{k-j}(g)$, and $D^{s-1-j} t^{s-1} = \big((s-1)!/j!\big)t^j$, we find that $P_T(t) = \sum_{j=0}^{s-1} \frac{1}{j!} \binom{s-1}{j} t^j D^j(f)$.

(a) $f(t) = P_R(t) = (1-t^3)\frac{1}{(1-t)^3} = \frac{1+t+t^2}{(1-t)^2}$. For $s = 2$, the Poincaré series of T is $f(t) + tf'(t) = \frac{(1-t^3)}{(1-t)^3} + t\frac{(1-t)^2(1+2t)-(1+t+t^2)(-2)(1-t)}{(1-t^2)^4} \frac{1-t^3}{(1-t)^3} + t\frac{(1-t)(1+2t)+2+2t+2t^2}{(1-t)^3} = \frac{1-t^3+t+t^2-2t^3+2t+2t^2+2t^3}{(1-t)^3} = \frac{1+3t+3t^2-t^3}{(1-t)^3}$.

(b) Here, $f(t) = (1-t)^{-r}$. Hence, $f^{(j)}(t) = (-r)(-r-1)\cdots(-r-(j-1))(-1)^j(1-t)^{-r-j} = r(r+1)\cdots(r+j-1)(1-t)^{-(r+j)}$. Hence, $P_T(t) = \sum_{j=0}^{s-1} \binom{r+j-1}{j}\binom{s-1}{j} t^j (1-t)^{-(r+j)} = \frac{\sum_{j=0}^{s-1} \binom{r+j-1}{j}\binom{s-1}{j} t^j (1-t)^{s-1-j}}{(1-t)^{r+s-1}}$. If $s = 2$ this is $\frac{1+(r-1)t}{(1-t)^{r+1}}$.

6. $\mathrm{Hilb}_{S^{(d)}}(n)$ is the number of dn-forms in s variables, which is $\binom{dn+s-1}{s-1}$. Note that $S^{(d)}$ is a direct summand of S as an $S^{(d)}$-module: the retraction fixes monomials of degree d and kills the other monomials. Since $x_1^d, \dots, x_s^d$ is a regular sequence in S, it is a regular sequence in $S^{(d)}$, and we may apply **#4**. Hence, $P(t) = P_{S^{(d)}}(t)$ will have denominator $(1-t)^s$. Let $Q_{s,d}$ be the polynomial in t such that the coefficient of t^i is the number of forms of degree di in s variables in which all exponents are at most $d-1$. The Poincaré series is $Q_{s,d}/(1-t)^s$. The highest degree in which there is such a form is $\lfloor((d-1)(s)/d)\rfloor = s - \lceil \frac{s}{d} \rceil$, where $\lceil r \rceil$ is the least integer $\geq r$. Hence, $\mathfrak{a}(S^{(d)}) = -\lceil \frac{s}{d} \rceil$. $Q_{s,d}$ can be calculated more explicitly as follows. The number of n-forms in which all exponents are degree at most $d-1$ is the Hilbert function of $K[x_1, \dots, x_s]/(x_1^d, \dots, x_s^d)$ and this is $(\Delta_d)^s \binom{n+s-1}{s-1}$ where Δ_d sends $g(n)$ to $g(n) - g(n-d)$. The sth iteration of Δ_d applied to g is $g(n) - \binom{s}{1}g(n-d) + \binom{s}{2}g(n-2d) + \cdots + (-1)^s\binom{s}{s}g(n-sd)$. Since g is 0 when evaluated at negative integers, when $n = di$, this is $\sum_{j=0}^{i}(-1)^j\binom{s}{j}g\bigl(d(i-j)\bigr) = \sum_{j=0}^{i}(-1)^j\binom{s}{j}\binom{d(i-j)+s-1}{s-1}$.

EXTRA CREDIT 3. By adjoining k_i th roots of generators of degree k_i, we may embed the ring in a ring generated by one forms. Hence, the Hilbert function of any $\mathbb{N}$-graded ring R with $\dim R = d$ is bounded by a polynomial of degree $d-1$. The Fibonacci numbers grow faster than any polynomial. Alternatively, the Poincaré series of a graded ring has a denominator all of whose roots are roots of unity. The given series satisfies $P = 1 + tP + t^2P$ and so $P = 1/(1-t-t^2)$. The roots of the denominator are $(-1 \pm \sqrt{5})/2$, and do not have absolute value 1.

EXTRA CREDIT 4. Let the degrees of the generators $f_1, \dots, f_k$ be $d_1, \dots, d_k$. Let L be the least common multiple of the degrees of the generators, and let $m_i = L/d_i$. If $n \geq N = (\sum_{i=1}^{k} m_i) - (k-1)$, any monomial in $f_1, \dots, f_k$ of degree n must be divisible by $f_i^{m_i}$ for some i, which is a monomial in the $f_1, \dots, f_k$ of degree L. Hence, for $n \geq N - L$, $(*)$ $R_{n+L} = R_L R_n$. Choose $d \geq N - L$ so that it is a multiple of L, say $d = aL$. Then $R_{hd} = (R_d)^h$ for all $h \geq 1$ by induction on h, since $R_{(h+1)d} = R_{hd+d} = R_{hd+aL} = R_L R_L \cdots R_L R_{hd}$ with a copies of R_L, using a iterations of $(*)$, and $(R_L)^a \subseteq R_{aL} = R_d$. $\square$