Math 615, Winter 2012 **Problem Set #3**

Due: March 16.

1. Let M and N be R-modules and let S be a flat R-algebra. Show that there is a natural isomorphism $S \otimes_R \operatorname{Tor}_i^R(M, N) \cong \operatorname{Tor}_i^S(S \otimes_R M, S \otimes_R N)$. In particular, the calculation of Tor commutes with localization and, if R is local and M, N are finitely generated, with completion.

2. Let M be a finitely generated module over a domain A. Let $u_1, \dots, u_r$ be a maximal set of elements in M that are linearly independent over A. Show that there exists $a \in A - \{0\}$ such that $aM \subseteq \sum_{i=1}^r Au_i = G$, which is free.

(a) Show that $\operatorname{Tor}_i^A(M, N)$ is killed by a for every $i \geq 1$ and every A-module N.

(b) Prove that M_a is free over A_a.

(c) Show that if $M \hookrightarrow Q$ as A-modules and $M \otimes_A N$ is torsion-free, then $M \otimes_A N \hookrightarrow Q \otimes N$ as A-modules.

3. Let M be a Cohen-Macaulay module over a local ring (R, m, K). Let P be a prime ideal such that $M_P \neq 0$. Replace R by $R/\operatorname{Ann}_R M$ and so assume that M is faithful. Note that M_P is faithful over R_P, since calculation of the annihilator commutes with localization.

(a) Show that there are elements $x_1, \dots, x_k$ in P whose images are a system of parameters for R_P and that are part of a system of parameters for R.

(b) Show that M_P is Cohen-Macaulay over R_P.

4. Let $R = K[[X, Y]]/(X^2, XY) = K[[x, y]]$, where K is a field, X, Y are formal indeterminates with imges x, y, and $m = (x, y)R$ is the maximal ideal.

(a) Prove that a minimal first module of syzygies of m is isomorphic with $K \oplus m$.

(b) Find all the minimal modules of syzygies of K, and determine the Betti numbers of $K = R/m$.

5. Let M be a finitely generated module over a local ring (R, m, K). Let $f_1, \dots, f_d \in R$ form a regular sequence on M. Prove that every nonzero submodule of M has Krull dimension at least d.

6. A Noetherian ring R is called *Cohen-Macaulay* if all of its localizations at prime ideals are Cohen-Macaulay. By **#3.(b)**, it suffices if the localizations at maximal ideals are Cohen-Macaulay. Prove that if R is Cohen-Macaulay, so is the polynomial ring $R[x_1, \dots, x_n]$ in n-variables over R. (Reduce to the case where $n = 1$, (R, m, K) is local, and $R[x]$ is localized at maximal ideal lying over m.)

EXTRA CREDIT 5. Let R be any ring. Show that an R-module M is flat if and only if $\operatorname{Tor}_1^R(R/I, M) = 0$ for every ideal $I \subseteq R$. Show that if R is Noetherian, M is flat iff $\operatorname{Tor}_1^R(R/P, M) = 0$ for every prime ideal P of R.

EXTRA CREDIT 6. Let $(R, m, K) \to (S, n, L)$ be a homomorphism of local rings such that m maps into n. Prove that S is R-flat if and only if $\operatorname{Tor}_1^R(K, S) = 0$.