Math 615, Winter 2012 **Problem Set #3: Solutions**

**1.** Let $G_\bullet$ be a projective resolution of $M$. Since $S$ is $R$-flat, $S \otimes_R G_\bullet$ is a projective resolution of $S \otimes_R M$. ($P \oplus Q \cong R^{\oplus \Lambda}$ yields $S \otimes_R P \oplus S \otimes_R Q \cong S^{\oplus \Lambda}$.) Since $S$ is flat, $S \otimes_R \_$ commutes with taking homology. (The calculation of homology $H_i$ of $C_\bullet$ is expressed by the short exact sequences $0 \to Z_i \to C_i \to B_{i-1} \to 0$ and $0 \to B_i \to Z_i \to H_i \to 0$ for all $i$: these are all preserved by $S \otimes_R \_$.) We have that $S \otimes_R \mathrm{Tor}_i^R(M, N) \cong S \otimes_R H_i(G_\bullet \otimes_R N) \cong H_i\big(S \otimes_R (G_\bullet \otimes_R N)\big)$. For any $R$-module $G$, $S \otimes_R (G \otimes_R N) \cong (S \otimes_S S) \otimes_R (G \otimes_R N) \cong (G \otimes_R S) \otimes_S (S \otimes_R N) \cong (S \otimes_R G) \otimes_S (S \otimes_R N)$ by repeated use of the commutativity and associativity of $\otimes$. Consequently, $H_i\big(S \otimes_R (G_\bullet \otimes_R N)\big) \cong H_i\big((S \otimes_R G_\bullet) \otimes_S (S \otimes_R N)\big) = \mathrm{Tor}_i^S(S \otimes_R M, S \otimes_R N)$. $\square$

**2.** Let $\mathcal{K} = \mathrm{frac}(A)$. Then $\dim_{\mathcal{K}} \mathcal{K} \otimes_R M = r$ is finite: pick $u_1, \ldots, u_r$ in $M$ whose images in $\mathcal{K} \otimes_R M$ form a $\mathcal{K}$-vector space basis. For each of the finite set of generators $m_j$ of $M$, there exists $a_j \in A - \{0\}$ with $a_j m_j \in G = Au_1 + \cdots + Au_r \cong Au_1 \oplus \cdots \oplus Au_r \cong A^{\oplus r}$. Let $a$ be the product of the $a_j$. Then $aM \subseteq G$, i.e., $M/G$ is killed by $a$.

(a) From the long exact sequence for Tor applied to $0 \to G \to M \to A/G \to 0$, for every $i \geq 1$ we have $\mathrm{Tor}_i^A(G, N) \to \mathrm{Tor}_i^A(M, N) \to \mathrm{Tor}_i^A(M/G, N)$. Since $G$ is $A$-free (it may be 0), we have that $\mathrm{Tor}_i^A(G, N) = 0$ and so $\mathrm{Tor}_i^A(M, N)$ injects into $\mathrm{Tor}_i^A(M/G, N)$. Since $M/G$ is killed by $a$, so is $\mathrm{Tor}_i^A(M/G, N)$.

(b) When we localize the sequence $0 \to G \to M \to M/G \to 0$ at the element $a$, we have that $(M/G)_a = 0$, and so $G_a \cong M_a$ is free over $A_a$.

(c) By the long exact sequence for Tor, we have $T = \mathrm{Tor}_1^A(Q/M, N) \to M \otimes_A N \to Q \otimes_A N$, and it suffices to show that the image of the leftmost module is 0. If $Q/M$ is finitely generated, this follows from part (a) because $T$ is killed by some $a \in A - \{0\}$. In general, because $Q/M$ is a directed union of finitely generated modules, every element of $\mathrm{Tor}_1^A(Q/M, N)$ is killed some $a \in A - \{0\}$, and so the the image is 0 in general, since $M \otimes_A N$ is torsion-free over $A$.

**3.** We assume that $M$ is faithful.

(a) To obtain a first element $x_1 \in P$ that will be part of a system of parameters in $P$ and of $R$, it suffices to choose an element of $P$ that is not in any minimal prime of $R$ (its image is then automatically not in any minimal prime of $R_P$). If $P$ were contained in the union of the minimal primes of $R$, by the prime avoidance lemma it would be contained in one of them, hence, minimal, i.e., of height 0, and the empty set of parameters may then be used. Otherwise, we have an element of $P$ not in any minimal prime of $R$. This will be the first element of system of parameters for $R$, and its image a parameter for $R_P$. The problem of constructing the rest of the system of parameters may then be solved for $R/x_1R$ and $P/x_1R$. It follows by induction on the height of $P$ that we can construct the system. $\square$

(b) Given such a system, it is a regular sequence on $M$, and localization preserves this property. (Localization is flat: hence, the property of being a nonzerodivisor is preserved. By a straightforward induction, the property of being a possibly improper regular sequence is preserved.) Since the $x_i$ are in $P$ and $M_P \neq 0$ we have that $M_P/PM_P \neq 0$.

**4.** $(x, y) = m = \mathrm{syz}^1 K$, and we want to describe the set $\{(r, s) \in R^2 : rx + sy = 0\}$. We must have $r \in m$, and then it is automatic that $rx = 0$. In this case we must have $sy = 0$, and this implies that $s \in xR \cong K$, since it is killed by $m$. It follows that $\mathrm{syz}^1 m \cong m \oplus K$. By a straightforward induction, $\mathrm{syz}^n(K) \cong m^{f_n} \oplus K^{f_{n-1}}$, where $f_0,\ f_1, f_2,\ \ldots$ is the Fibonacci sequence $0,\ 1,\ 1,\ 2,\ 3,\ 5,\ \ldots$. Since $m$ needs two minimal generators and $K$ needs one, the $n$ th Betti number is $2f_n + f_{n-1} = f_n + (f_n + f_{n-1}) = f_n + f_{n+1} = f_{n+2}$. $\square$

**5.** We use induction on $\dim(M)$. If $d = 0$ there is nothing to prove. If $d = 1$ and $x$ is a nonzerodivisor on $M$, it is a nonzero divisor on every proper nonzero submodule, and so the submodule cannot have dimension 0 (for then it would be killed by a power of $m$). Thus, we may assume that $d \geq 2$. Suppose that $M$ has a nonzero submodule $N$ of dimension at most $d-1$. Let $f = f_1$ and let $N_i = N :_M f_1^i$. This is an ascending sequence of submodules of $M$ containing $N_1$. Hence, it stabilizes. Suppose that $N :_M f^t = N :_M f^{t+1} = \cdots$. Then, since $f$ is a nonzerodivisor on $M$, $N_t \cong f^t N_t \subseteq N$, and $\dim(N_t) \leq d-1$ as well. By construction $N_t :_M f = N_t$, and $f$ is a nonzero divisor on every term of the short exact sequence $0 \to N_t \to M \to M/N_t \to 0$. Hence, we get a short exact sequence $0 \to N_t/fN_t \to M/fM \to (M/N_t)/f(M/N_t) \to 0$. We have that $f_2,\ \ldots,\ f_d$ is a regular sequence on $M/fM$, and it follows that $\dim(N) - 1 = \dim(N_t/fN_t) \geq d-1$, and so $\dim(N) \geq d$ after all, as required. $\square$

**6.** The reduction to the case $n = 1$ is clear. Let $x = x_1$ and $Q \subseteq R[x]$ be prime lying over $P$ in $R$. We may replace $R$ by $R_P$. Thus, it suffices to show that if $(R, m, K)$ is Cohen-Macaulay and $Q$ is a maximal ideal of $R[x]$ lying over $m$, then $R[x]_Q$ is Cohen-Macaulay. Note that $R \to R_Q$ is flat. A regular sequence in $m$ is therefore regular in $R[x]_Q$ and we may kill it in both rings without affecting the Cohen-Macaulay property: we may assume $\dim(R) = 0$. If $Q = mR[x]$, then $\dim(R[x]_Q) = 0$ and we are done. In the remaining case, $Q$ is generated mod $mR[x]$ by a monic polynomial $f$ whose image in $K[x]$ is irreducible, so that it generates a maximal ideal. In this case, $\dim R_Q = 1$, and $f$ is a parameter. Since $f$ is monic, it is a nonzerodivisor in $R[x]$ and, hence, in $R[x]_Q$. $\square$

**EXTRA CREDIT 5.** It suffices to show $\_\otimes_R M$ preserves inclusions of finitely generated modules, since every inclusion is a directed union of inclusions of finitely generated modules. Since every finitely generated module has a finite filtration with factors of the form $R/I$ (where $I$ can be taken to be prime in the Noetherian case), it suffices to show that if $\mathrm{Tor}_1(N_i,\ M) = 0$ for all the factors $N_i$ of a finite filtration of $N$, the same holds for $N$. (Then apply the long exact sequence for Tor to $0 \to N' \hookrightarrow N'' \to N \to 0$.) This follows by a straightforward induction on the length of the filtration and the long exact sequence for Tor applied to $0 \to N_1 \to N \to N/N_1 \to 0$.

**EXTRA CREDIT 6.** "Only if" is clear. As in **EC #5**, it suffices to show that inclusions $N \hookrightarrow M$ are preserved for finitely generated modules. Suppose that $S \otimes_R N \to S \otimes_R M$ has a nonzero kernel $Z$. Then we can choose $t \gg 0$ such that $Z$ is not contained in $n^t(S \otimes_R N)$, and, hence, not in $m^t(S \otimes_R N)$, which is the image of $S \otimes m^t N$. By the Artin-Rees lemma we can choose $k$ so large that $m^k M \cap N \subseteq m^t N$. Let $\overline{M} = M/m^k M$ and $\overline{N} = N/(m^k M \cap N)$. Then $\overline{N} \subseteq \overline{M}$ have finite length, the image of $Z$ in $S \otimes_R \overline{N}$ is not 0, and, hence, $S \otimes_R \overline{N} \to S \otimes_R \overline{M}$ has a nonzero kernel. But $C = \overline{M}/\overline{N}$ has finite length and has a filtration with finitely many copies of $K$ as factors. It follows as in **EC #5** that $\mathrm{Tor}_1^R(C,\ S) = 0$, and so $S \otimes \overline{N} \to S \otimes \overline{M}$ is injective using the long exact sequence for Tor, a contradiction. $\square$