Math 615, Winter 2012 **Problem Set #4**

Due: Monday, April 2.

1. Let M and N be finitely generated modules over a PID R, and let A, B be their respective torsion submodules. Show that $\mathrm{Tor}_1^R(M, N) \cong A \otimes_R B$ (not canonically.)

2. Let R be a ring and f, g nonzerodivisors.

Calculate (a) $\mathrm{Ext}^1(R/fR, R)$ and also (b) $\mathrm{Ext}^1(R/fR, R/gR)$ in these three cases:

(1) f, g is a regular sequence in R. (2) g is a multiple of f. (3) f is a multiple of g.

Use your results to discuss the behavior of $\mathrm{Ext}^1_R(M, N)$ where M and N are finitely generated modules over a principal ideal domain.

3. Let $S = K[X_1, \ldots, X_n]$ be a polynomial ring in n variables over a field K and let $F = X_1 \cdots X_n$ be the product of the variables. Let $R = S/FS$. Show that $G_0(R) \cong \mathbb{Z}^n$.

4. Let $X = \mathrm{Spec}\,(R)$, where R is Noetherian. Show that $U \subseteq X$ is open if and only if it satisfies the following two conditions:

(a) If $Q \in U$ and $P \subseteq Q$, then $P \in U$.

(b) For all $Q \in U$, $V(Q) \cap U$ contains an open neighborhood of Q in $V(Q)$.

[Together with results in class, this will show, for example, that if R is a finitely generated algebra over an algebraically closed field, the set $\{P \in \mathrm{Spec}\,(R) : R_P \text{ is regular}\}$ is open.]

5. Let M be a Cohen-Macaulay module of Krull dimension d over a regular local ring (R, m, K) of dimension n. Let $G_\bullet$ denote a minimal free resolution of M over R. Show that $\mathrm{Hom}_R(G_\bullet, R)$ has no homology except in degree $n - d$, and consequently, numbered backwards gives a minimal free resolution of $M^\vee = \mathrm{Ext}^{n-d}(M, R)$. Show as well that $M^\vee$ is Cohen-Macaulay of Krull dimension d, and that $(M^\vee)^\vee \cong M$. Conclude that $\_^\vee$ is a contravariant exact functor from Cohen-Macaulay modules of Krull dimension d over R to themselves, and show as well that $\mathrm{Ann}_R M = \mathrm{Ann}_R M^\vee$.

6. Let A and B be given modules over a ring R. Show that the isomorphism classes of short exact sequences $0 \to A \to M \to B \to 0$ are in bijective correspondence with the elements of $\mathrm{Ext}^1_R(B, A)$. [Apply $\mathrm{Hom}_R(B, \_)$. In the long exact sequence the image of id_B is in $\mathrm{Ext}^1_R(B, A)$. Also, If $G_\bullet$ is a projective resolution of B and $h : G_1 \to A$ represents an element of $\mathrm{Ext}^1_R(B, A)$, h induces $f : B_1 \to A$, where $B_1 = \mathrm{Ker}\,(G_0 \twoheadrightarrow B)$, and one may construct M as $(A \oplus G_0)/\{f(b) \oplus -b : b \in B_1\}$.]

EXTRA CREDIT 7. Let K be a field and let $R = K[X, Y, U, V]/(XU - YV] = K[x, y, u, v]$. Show that $G_0(R)$ is generated by $[R]$ and $[R/P]$, where $P = (x, y)R$. Show moreover, that $G_0(R) \cong \mathbb{Z} \oplus \mathbb{Z}$.

EXTRA CREDIT 8. Let $(*)$ $\;0 \to A \to B \to C \to 0$ be a short exact sequence of finitely generated modules over a Noetherian ring R. Show that if $B \cong A \oplus C$, possibly in some other way, then the sequence $(*)$ must be split. [Suggestion: First do the case where the three modules have finite length.]