Math 615, Winter 2012 **Problem Set #4: Solutions**

1. Each of M, N is the direct sum of a free module and a torsion module: say $M = F \oplus A$ and $N = G \oplus B$. Since Tor distributes over $\oplus$ and higher Tors with free modules are 0, $\mathrm{Tor}_1^R(M,\, N) \cong \mathrm{Tor}_1^R(A,\, B)$. Thus, it suffices to show that $\mathrm{Tor}_1^R(A, B) \cong A \otimes_R B$. A and B are finite direct sums of cyclic torsion modules. Since both $\mathrm{Tor}^R(\_\,, \_)$ and $\otimes_R$ distribute over $\oplus$, we may assume that $A = R/aR$, $B = R/bR$, $a, b \neq 0$. Then $A \otimes_R B \cong R/(a,b)R = R/dR$ where $d =$GCD(a,b). Use the resolution $0 \to R \xrightarrow{a} R \to 0$ for R/aR to see that $\mathrm{Tor}_R^1(R/aR, R/bR) = \mathrm{Ker}\,(R/bR \xrightarrow{a} R/bR)$. If $a = a'd$, $b = b'd$ where $\mathrm{GCD}(a', b') = 1$, the kernel is $b'R/bR$ (since $\mathrm{GCD}(a', b') = 1) \cong b'R/b'dR \cong R/dR = R/(a,b)R$.

2. If M is any R-module and f is a nonzerodivisor in R, $0 \to R \xrightarrow{f} R \to 0$ resolves R/fR, and so $\mathrm{Ext}_R^1(R/fR,\, M)$, computed by applying $\mathrm{Hom}_R(\_\,,\, M)$ is $\mathrm{Coker}\,(M \xrightarrow{f} M) \cong M/fM$. Hence, $\mathrm{Ext}^1(R/fR,\, R) \cong R/fR$, and $\mathrm{Ext}_R^1(R/fR,\, R/gR) \cong R/(f,g)R$ in all cases. Now suppose that $M \cong F \oplus A$ and $N \cong G \oplus B$ over the PID R as in Problem **#1.** Then $\mathrm{Ext}_R^1(F,\, \_)$ vanishes, and so $\mathrm{Ext}_R^1(M, N) \cong \mathrm{Ext}_R^1(A, N)$. For each cyclic summand R/fR of A, $\mathrm{Ext}_R^1(R/fR,\, N) \cong N/fN \cong (R/fR) \otimes_R N$. Hence, $\mathrm{Ext}_R^1(M, N) \cong A \otimes_R N$.

3. Every prime in R contains one of the X_i and so is in the image of some $G_0(R/X_iR) \to G_0(R)$. But $G_0(R/X_iR) \cong \mathbb{Z}$, where $[R/X_iR]$ generates. It follows that the n elements $[R/X_iR]$ span $G_0(R)$. Now suppose that $\sum_{i=1}^n a_i[R/X_iR] = 0$, where the $a_i \in \mathbb{Z}$. To complete the proof, it suffices to show that all the a_i are 0. Let $P_i = X_iR$. The map $M \mapsto \mathrm{length}_{R_{P_i}}(M_{P_i})$ is additive on short exact sequences and so induces a map $G_0(R) \to \mathbb{Z}$ that sends $[R/P_j] \mapsto 0$ if $j \neq i$ and $[R/P_i] \mapsto 1$. Hence, its value on $\sum_{i=1}^n a_i[R/X_iR]$ is a_i. Consequently, if the sum is 0, all the a_i are 0. $\square$

4. Necessity is clear. We use Noetherian induction on R: we may assume the result for all R/I, $I \neq 0$. Let $P_1, \ldots, P_k$ be the minimal primes of R. Then $\mathrm{Spec}\,(R) = \bigcap_i V(P_i)$, and so it suffices to show that $U_i = U \cap V(P_i)$ is open in $V(P_i)$ for all i. U_i satisfies the same conditions within $V(P_i) = \mathrm{Spec}\,(R/P_i)$. So we have reduced to the case of one minimal prime, P. If $P \notin U$, U must be empty. Otherwise, choose $f \notin P$ such that the open set $D_f \subseteq U$. It suffices to show that $U \cap V(f)$ is open in $V(f)$. But we may identify $\mathrm{Spec}\,(R/fR)$ with $V(f)$, and then $U \cap V(f)$ satisfies the same conditions. By the hypothesis of Noetherian induction, $U \cap V(f)$ is open in $V(f)$. $\square$

5. Note that $\dim M = d$ implies that $I = \mathrm{Ann}_R M$ has height $n-d$. Let $x_1, \ldots, x_h = \underline{x}$ be a maximal regular sequence on R in I, and let $P \supseteq I$ be an associated prime of $(\underline{x})$. Then $\underline{x}$ is a maximal regular sequence in R_P, which is Cohen-Macauly, so P has height $h \geq$ height I. Let Q be a minimal prime of I of height $n-d/$ Any regular sequence in I is part of a system of parameters for R_Q, so $h \leq n-d$. Thus, $h = n-d$, and $\mathrm{depth}_I\, R = n-d$. By a class theorem, the first nonvanishing $\mathrm{Ext}_R^j(M, R)$ occurs with $j = n-d$. Since M is Cohen-Macaulay, by the Auslander-Buchsbaum theorem $\mathrm{pd}_R M = \mathrm{depth}\,(R) - \mathrm{depth}\,(M) = n - \dim(M) = n-d$. Hence, there is a unique nonvanishing $\mathrm{Ext}_R^j(M, R)$ for $j = n-d$. Since $\mathrm{Ext}_R^\bullet(M, R) = H^\bullet(\mathrm{Hom}_R(G_\bullet, R))$, $\mathrm{Hom}_R(G_\bullet, R)$, numbered backwards, is a free resolution of $\mathrm{Ext}_R^{n-d}(M, R) = M^\vee$. The matrices for the dual bases in the dual complex are the transposes of those in the original complex, and so have entries in m. Hence, this resolution of $M^\vee$ is minimal. If we use the dual complex to calculate $(M^\vee)^\vee = \mathrm{Ext}_R^{n-d}(M^\vee, R)$ we get the original complex back, and so $M^{\vee\vee} \cong M$. Clearly, $\mathrm{Ann}_R M$ kills $M^\vee$, and $\mathrm{Ann}_R M^\vee$ kills $(M^\vee)^\vee$. Hence, M and $M^\vee$ have the same annihilator and the same dimension. Also $\mathrm{pd}(M^\vee) = n - d$ implies that $\mathrm{depth}\,(M^\vee) = d$. Thus, $M^\vee$ is Cohen-Macaulay, and $M^{\vee\vee} \cong M$. Exactness is immediate from the long exact sequence for Ext (only the terms in degree $n-d$ are nonzero). Note also that if x is a nonzerodivisor on M, the long exact sequence for Ext gives that x is not a zerodivisor on $M^\vee$ and an isomorphism $(M/xM)^\vee \cong M^\vee/xM^\vee$, which yields a different proof, by induction on $\dim(M)$, that $M^\vee$ is Cohen-Macaulay.

6. Given the short exact sequence $0 \to A \to M \to B \to 0$, denoted $\mathcal{S}$, we get a map of $G_\bullet$ to the complex $0 \to A \to M \to 0$ that lifts id_B. The map $h : G_1 \to A$ must kill $\mathrm{Im}\, G_2$ and so gives a map $f : B_1 \to A$. Every such f in fact arises from a unique $h : G_1 \to A$ that kills $\mathrm{Im}\, G_2$. The maps $A \hookrightarrow M$ and $G_0 \twoheadrightarrow M$ give a map $A \oplus G_0 \twoheadrightarrow M$ and $a \oplus b$ maps to 0 iff $b \in B_1$ and $a + f(b_1) = 0$. Thus $0 \to A \to M \to B \to 0$ arises, up to isomorphism, from $f : B_1 \to A$ by the construction given: with $N_f = \{(f(b_1) \oplus -b_1 : b_1 \in B_1\}$, the constructed module in the middle is $M_f = (A \oplus G_0)/N_f$. To get the bijection with $\mathrm{Ext}^1_R(A, B)$ it suffices to show that $f, f' : B_1 \to A$ give isomorphic extensions iff $f - f'$ extends to G_0. For "if", note that if ϕ extends $f' - f$ to G_0 the map $A \oplus G_0 \to A \oplus G_0$ given by $a \oplus u \mapsto (a + \phi(u)) \oplus u$ induces the isomorphism. For "only if", we may use the isomorphism of sequences to get a map $\theta : A \oplus G_0 \to A \oplus G_0$ that is the identity on A and induces the identity map on B, which means we may take the map on G_0 to have the form $\phi \oplus \mathrm{id}_{G_0}$, where $\phi : G_0 \to A$. For θ to take Z_f to $Z_{f'}$, as needed, we must have that ϕ extends $f' - f$ to all of G_0. We have at once from the construction of the connecting homomorphism (using the snake lemma on the short exact sequence of complexes obtained by taking $\mathrm{Hom}_R(G_\bullet, \mathcal{S})$) that id_B maps to $[h] \in \mathrm{Ext}^1(B, A)$: id_B is induced by the map $G_0 \to M$, and composing with $G_1 \to G_0$ gives h, which actually takes values in A.

EXTRA CREDIT 7. Since $G_0(R/xR) \to G_0(R) \to G_0(R_x) \to 0$ is exact, $G_0(R)$ will be generated by $\mathrm{Im}\, G_0(R/xR)$ and lifts of generators of $G_0(R_x)$. Since $R_x \cong K[x,y,z]_x$ (we may solve for u) and $G_0(K[x,y,z]) \cong \mathbb{Z}$, generated by the class of the ring, we have that $G_0(R)$ is generated by $[R]$ (lifting $[R_x]$) and $\mathrm{Im}\, G_0(R/xR)$. Now, $R/xR \cong (K[y,z]/(yz))[u]$, polynomial over $K[y,z]/(yz)$. Hence, $G_0(R/xR) \cong G_0(K[yz]/(yz)) \cong \mathbb{Z} \oplus \mathbb{Z}$ generated by $[R/(x,y)]$ and $[R/(x,z)]$ by Problem **#3.** But $[R/xR] = 0$ in $G_0(R)$, and $0 \to R/(x,y) \xrightarrow{z} R/xR \to R/(x,z) \to 0$ is exact, so that $[R/(x,z)] = -[R/(x,y)]$ in $G_0(R)$. Hence, $[R]$ and $[R/(x,y)]$ generate $G_0(R)$. Since R is a domain, $\mathbb{Z} \cdot [R]$ splits off. We need only show that $[R/(x,y)]$ is not torsion in $\overline{G}_0(R)$. But $P = (x,y)R$ has infinite order in the divisor class group of R: note that $R \cong K[as, at, bs, bt]$ with as, at, bs, bt corresponding to x, y, u, v respectively, that $(x,y)^k$ is the contraction of the primary ideal a^k in $K[a,b,s,t]$, hence primary, and so $P^{(k)} = P^k$, which needs $k+1$ minimal generators. $\square$

EXTRA CREDIT 8. Finite length case. We have $0 \to \mathrm{Hom}_R(C, A) \to \mathrm{Hom}_R(C, B) \to \mathrm{Hom}_R(C, C) \to D \to 0$. Because $B \cong A \oplus C$ in some way, $\mathrm{length}\, \mathrm{Hom}_R(C, B)$ is the sum of the lengths of $\mathrm{Hom}_R(C, A)$ and $\mathrm{Hom}_R(C, C)$. Hence, $\mathrm{length}\, D = 0$, and so $D = 0$. But then id_C is the image of a map $C \to B$, which means that the sequence splits. We leave the general case as a continuing problem. $\square$