Math 615, Winter 2012 **Problem Set #5**

Due: Monday, April 16.

1. Let M be a Cohen-Macaulay module of dimension d over a local ring $(R,\, m,\, K)$. Let $\underline{x} = x_1,\, \dots\,, x_d$ be a system of parameters (i.e., a maximum regular sequence) on M, and let $x = x_1$, so that x is a nonzerodivisor on M.

(a) Prove that $\mathrm{Ext}^d_R(K,\, M) \cong \mathrm{Ext}^{d-1}_R(K,\, M/xM)$.

(b) Prove that $\mathrm{Ext}^d_R(K,\, M) \cong \mathrm{Hom}_R\big(K,\, M/(\underline{x})M\big)$. Hence, the K-vector space dimension of $\mathrm{Hom}_R\big(K,\, M/(\underline{x})M\big) \cong \mathrm{Ann}_{M/(\underline{x})M}m$ is independent of the choice of system of parameters $x_1,\, \dots\,, x_d$.

The positive integer $\dim_K \mathrm{Ext}^d_R(K,\, M)$ is called the *type* of M. Also show that the type of M is the same as the type of $\widehat{M}$ over $\widehat{R}$.

2. A local ring (R, m, K) that has type 1 as a module over itself is called *Gorenstein.* Prove if R is regular, then R is Gorenstein, and that if R is Gorenstein, so is $R/(f_1,\, \dots\,, f_k)R$ whenever $f_1,\, \dots\,, f_k$ is part of a system of parameters for R.

3. Let M be a Cohen-Macaulay module of dimension d over a regular local ring $(R,\, m,\, K)$ of dimension n. Show that the type of M is the same as the least number of generators of its Ext dual $\mathrm{Ext}^{n-d}_R(M,\, R)$. (It may be helpful to reduce to the case where Krull dim $M = 0$.)

4. Let $X = (x_{ij})$ denote a 3×2 matrix of indeterminates over a field K, and let R be the polynomial ring in the six variables x_{ij} over the field K. Let m denote the ideal generated by the variables. Let Δ_1, $-\Delta_2$ and Δ_3 be the determinants of the 2×2 matrices obtained by omitting the first, second and third rows of the matrix, respectively, and let Y be the 1×3 matrix $(\Delta_1\ \Delta_2\ \Delta_3)$, So that $YX = (0)$. Let $P = (\Delta_1, \Delta_2, \Delta_3)R$. You may assume that the complex $\quad (*) \quad 0 \to R^2 \xrightarrow{X} R^3 \xrightarrow{Y} R \to R/P \to 0 \quad$ is exact, and so gives a free resolution of R/P. Let Q be the ideal generated by x_{12}, $x_{11} - x_{22}$, $x_{21} - x_{32}$, and x_{31} in R. Show that the images of these elements form a homogeneous system of parameters for R/P, determine the type of R_m/PR_m in two different ways, and determine the intersection multiplicity of $Z = V(P)$ and $L = V(Q)$ at the origin.

5. Let $R = K[[x, y]]/(xy)$, where K is a field. Determine the minimal first modules of syzygies of R/xR and R/yR. Describe a minimal free resolution of R/xR over R and determine all the Betti numbers of R/xR over R. Find $\mathrm{Tor}^R_i(R/xR,\, R/yR)$ for all $i \geq 0$.

6. Let $(R, m, K) \to (S, n, L)$ be a flat local extension such that $\dim S/mS = 0$. Let M be a Cohen-Macaulay module over R. Show $S \otimes_R M$ is Cohen-Macaulay and its type is the product of the type of M and the type of S/mS.

EXTRA CREDIT 9. Prove that the complex described in **#4.** is exact.

EXTRA CREDIT 10. Use the resolution in **#4.** to calculate the Hilbert function of R/P. Show that R/P maps as K-algebra onto the Segre product T of the polynomial rings $K[x,\, y,\, z]$ and $K[s,\, t]$ so as to preserve degree. The Hilbert function of T was calculated in an earlier exercise. Conclude from the fact that R/P and T have the same Hilbert function that the map $R/P \twoheadrightarrow T$ is an isomorphsim.