Math 615, Winter 2012 **Problem Set #5: Solutions**

**1.** (a) By a class theorem, the first non-vanishing $\mathrm{Ext}^i_R(K, M)$ occurs at the depth of $M$ on $m$. Thus, $\mathrm{Ext}^{d-1}_R(K, M) = 0$. Therefore, the long exact sequence for Ext coming from $0 \to M \xrightarrow{x} M \to M/xM \to 0$ yields $0 \to \mathrm{Ext}^{d-1}_R(K, M/xM) \xrightarrow{\theta} \mathrm{Ext}^d_R(K, M) \xrightarrow{x} \mathrm{Ext}^d_R(K, M)$. The rightmost map is 0 because $x$ kills $K$, and so $\theta$ is an isomorphism. $\square$
(b) By a straightforward induction $k$, it then follows that for $1 \le k \le d$ that $\mathrm{Ext}^d_R(K, M) \cong \mathrm{Ext}^{d-k}_R(K, M/(x_1, \dots, x_k)M)$. The stated result is the case $k = d$. $\square$

Since $\widehat{R}$ is flat over $R$, completion commutes with Ext for finitely generated $R$-modules, from which the final statement follows.

**2.** If $R$ is regular, we may compute the type as the $K$-vector space dimension of $R/(\underline{x})R = K$, where $\underline{x} = x_1, \dots, x_d$ is a minimal set of generators of $m$ (but also, since $R$ is regular, a system of parameters. The result follows. By **1(b).**, type does not change when we kill part of a system of parameters: it can simply be computed for both rings after killing the rest of the system of parameters.

**3.** Replacing $M$ by $M/(x_1, \dots, x_k)M$ does not change the type nor the minimal number of generators, and replaces $M^\vee$ by $M^\vee/(x_1, \dots, x_k)M^\vee$. Thus, it suffices to consider the case where $M$ has finite length. Let $x_1, \dots, x_d$ generate the maximal ideal of $R$, and consider the map $M \to M^d$ sending $m \mapsto (x_1 m, \dots, x_d m)$. The kernel $V$ is a $K$-vector space and is evidently $\mathrm{Ann}_M m$. Since $(*)\quad 0 \to V \to M \to M^d$ is an exact sequence of 0-dimensional Cohen-Macaulay modules, applying the exact contravariant functor $\_^\vee$ yields an exact sequence $(M^\vee)^d \to M \to V^\vee \to 0$, where the leftmost map sends $(u_1, \dots, u_d)$ to $\sum_{i=1}^d x_i u_i$. It follows that $V^\vee \cong M^\vee/mM^\vee$. By Nakayama's lemma, the $K$-vector space dimension of the latter is the least number of generators of $M^\vee$. Thus, the result follows if $V$ and $V^\vee$ have the same dimension. Since $\_^\vee$ commutes with direct sum, it suffices to check this when $V = K$. But $\mathrm{Ext}^d(K, R/(x_1, \dots, x_d))$ has $K$-vector space dimension equal to the type of $R$, which is 1.

**4.** After killing the elements $\underline{f} = f_1, f_2, f_3, f_4$ suggested as a homogeneous system of parameters, the matrix has the form $\begin{pmatrix} u & 0 \\ v & u \\ 0 & v \end{pmatrix}$ where $u$ is the common image of $x_{11}$ and $x_{22}$ and $v$ is the common image of $x_{21}$ and $x_{32}$. The $2 \times 2$ minors are $v^2, uv, u^2$. Thus, the quotient $B = \cong K[u,v]/(u^2, uv, v^2) \cong K + Ku + Kv$ is Artin. Hence, $\dim (R/P) \le 4$. But $R$ maps onto the Segre product described in **EC10**, whose fraction field is $K(xs, ys, zs, t/s)$, which has transcendence degree 4. So $\dim R/P = 4$. From the given projective resolution, which has length 2, the depth of $(R/P)_m$ is 6-2 $= 4$. Thus $(R/P)_m$ is Cohen-Macaulay, and the given homogeneous system of parameters is a regular sequence. From the calculation of the quotient as $K + Ku + Kv$, the dimension of the annihilator of $m$ in the quotient, which is $Ku + Kv$, is 2. So the type is 2. This also follows from the fact that when uses the given resolution to compute $\mathrm{Ext}^2(R/P, R) \cong \mathrm{Coker}\, X^{tr}$, it needs two minimal generators, even after localization at $m$. Thus, the homogeneous system of parameters is a regular sequence. Finally, the required intersection multiplicity $e$ may be computed from the Koszul homology $H_i(\underline{f}; (R/P)_m)$. All of this homology vanishes for $i \geq 1$ since $\underline{f}$ is a regular sequence. Hence, $e$ is the length of $(R/P)_m/(\underline{f}) \cong K + Ku + Kv$, and so is 3.

**5.** The first module of syzygies of $R/xR$ is $xR \cong R/yR$. One has symmetry here. Hence, the minimal free resolution is $\cdots \xrightarrow{y} R \xrightarrow{x} R \xrightarrow{y} R \xrightarrow{x} R \to R/xR \to 0$: it is periodic with period 2, the maps are alternately multiplication by $x$ and multiplication by $y$, and all the Betti numbers are 1. When we omit the augmentation, apply $\otimes_R R/yR$, note that $R/yR = K[[x]]$, and that multiplication by $y$ becomes the 0 map, we obtain the Tors as the homology of $\cdots \xrightarrow{0} K[[x]] \xrightarrow{x} K[[x]] \xrightarrow{0} K[[x]] \xrightarrow{x} K[[x]] \to 0$. It follows that $\mathrm{Tor}_i^R(R/xR, R/yR) \cong K$ if $i$ is even and is 0 if $i$ is odd. (Tor is not rigid in this example.)

**6.** We may replace $R$ by $R/\mathrm{Ann}_R M$ and $S$ by $S/(\mathrm{Ann}_R M)S$. Let $y_1, \ldots, y_h$ be a system of parameters in $R$. We may replace $M$ by $M/(y_1, \ldots, y_h)M$. Since $S$ is $R$-flat, the $y_i$ form a regular sequence on $S \otimes_R M$ as well as on $M$. Since $\dim S/mS = 0$, $n$ is nilpotent mod $m$ and so it is nilpotent mod $(y_1, \ldots, y_h)S$. Thus, we may assume that $R$, $S$ are Artin local. Let $(x_1, \ldots, x_d) = m$. Then we have an exact sequence $(*)$ $\quad 0 \to V \to M \to M^d$ as in 3. above, where $V = \mathrm{Ann}_M m$, so that $\dim_K V = t$ is the type of $M$. Apply $S \otimes_R \_$ to obtain an exact sequence ($S$ is $R$-flat) $0 \to S \otimes_R V \to S \otimes_R M \to (S \otimes M)^d$. Then $\mathrm{Ann}_{S \otimes_R M)} m$ may be identified with $V \otimes_R S$, which, since $m$ kills $V$, may be identified with $N = V \otimes_K (S/mS)$, and $N$ contains the annihilator $N'$ of $n$ in $S \otimes_R M$. Hence, $N'$ may be identified with $\mathrm{Ann}_{V \otimes_K (S/mS)} n \cong V \otimes_K W$, with $W = \mathrm{Ann}_{S/mS} n$, an $L$-vector space with $\dim_L W = t'$, where $t'$ is the type of $S/mS$. Hence, $N' \cong V \otimes_K W \cong K^t \otimes_K L^{t'} \cong L^{tt'}$. $\square$

**EXTRA CREDIT 8., continued.** One needs that $\mathrm{Hom}_R(C, B) \to \mathrm{Hom}_R(C, C)$ is onto. The issue is local, and we may aslso complete. Thus, we may assume $(R, m)$ is complete local. For all $n$, $0 \to N \to A/m^n A \to B/m^n B \xrightarrow{g_n} C/m^n C \to 0$ is exact for a suitable kernel $N$. The hypothesis implies that the length of $B/m^n B$ is the sum of the lengths of the surrounding modules, which forces $N$ to be 0. For each $n$, there is a nonempty coset in $\mathrm{Hom}_R(C/m^n C, B/m^n B)$ consisting of maps $f$ such that $g_n \circ f = \mathrm{id}$, because we have already shown there are splittings in the finite length case. The inverse limit $W$ of these cosets is nonempty by a class lemma, and an element of $W$ induces a map of $C \to B$ that splits $B \twoheadrightarrow C$. $\square$

**EXTRA CREDIT 9.** The only issue that is not straightforward is exactness at the $R^3$ spot, which says that the columns $C_1, C_2$ of $X$ span the relations on the $\Delta_i$. Suppose that $f_1\Delta_1 + f_2\Delta_2 + f_3\Delta_3 = 0$. Then $\Delta_1, \Delta_2 \in (x_{31}, x_{32})$, which is prime, and $\Delta_3$ is not in this ideal. Hence, $f_3 = ux_{31} + vx_{32}$. It follows that if $f$ is the column given by the $f_i$, then $f - uC_1 - vC_2$ has third coordinate 0, so that it is a relation, essentially, on $\Delta_1$ and $\Delta_2$. Since $\Delta_1$ and $\Delta_2$ are relatively prime, this relation is a multiple of the relation $(-\Delta_2)\Delta_1 + (\Delta_1)\Delta_2 + (0)\Delta_3 = 0$, and the result follows because the column of coefficients in this relation is $x_{32}C_1 - x_{31}C_2$. $\square$

**EXTRA CREDIT 10.** The alternating sum of the Hilbert functions of the modules in the resolution is 0. Thus, the $\mathrm{Hilb}_{R/P}(n) = \binom{n+5}{5} - 3\binom{n+3}{5} + 2\binom{n+2}{5}$. Factoring $(n+2)(n+1)/5!$ from each term gives $(n+5)(n+4)(n+3) - 3(n+3)n(n-1) + 2n(n-1)(n-2) = (1-3+2)n^3 + (12-6-6)n^2 + (20+15+12+9+4)n + (60+0+0) = 60(n+1)$, so the Hilbert function is $(n+2)(n+1)^2/2 = \binom{n+2}{2}(n+1)$. $\square$ (Cf. Problem Set #2, **5(a).)**