Cohen-Macaulay rings

A sequence of elements $x_1, \ldots, x_n$ of a ring R is called a *possibly improper regular sequence* on the R-module M if x_1 is not a zerodivisor on M and, for $1 \le i \le n-1$, x_{i+1} is not a zerodivisor on $M/(x_1, \ldots, x_i)M$. If, in addition, $(x_1, \ldots, x_n)M \neq M$, the sequence is called a *regular sequence* on M. The main case here will be when $M = R$.

A local ring is called *Cohen-Macaulay* if some system of parameters is a regular sequence on R. We shall see that this implies that every system of parameters is a regular sequence. Note that a regular sequence $x_1, \ldots, x_n$ in a Noetherian ring R has image that is part of a system of parameters in every local ring R_P of R for a prime P with $(x_1, \ldots, x_n)R \subseteq P$. To see this, first note that the property of being a possibly improper regular sequence is preserved by flat base change, since that is true for the nonzerodivisor property: the injectivity of $R \xrightarrow{x\cdot} R$ is preserved when we apply $S \otimes_R \_$ to obtain $S \xrightarrow{x\cdot} S$, and the result follows from a straightforward induction on the number elements in the sequence. The condition $(x_1, \ldots, x_n) \subseteq P$ guarantees that one has a regular sequence in R_P. In the local case, if x_1 is not a zerodivsor, it is not in any minimal prime of R, and so $\dim(R/x_1R) = \dim(R) - 1$. The result that the regular sequence $x_1, \ldots, x_n$ is part of a system of parameters follows from a straightforward induction on n.

Regular local rings are Cohen-Macaulay: if one has a minimal set of generators of the maximal ideal, the quotient by each in turn is again regular and so is a domain, and hence every element is a nonzerodivisor modulo the ideal generated by its predecessors. Moreover, *local complete intersections*, i.e., local rings of the form $R/(f_1, \ldots, f_h)$ where R is regular and $f_1, \ldots, f_h$ is part of a system of parameters for R, are Cohen-Macaulay. It is quite easy to see that if R is Cohen-Macaulay, so is R/I whenever I is generated by a regular sequence.

If R is a Cohen-Macaulay local ring, we shall show below that the localization of R at any prime ideal is Cohen-Macaulay. We define an arbitrary Noetherian ring to be *Cohen-Macaulay* if all of its local rings at maximal ideals (equivalently, at prime ideals) are Cohen-Macaulay.

In general, regular sequences in a ring R are not permutable (e.g., x, $(1-x)y$, $(1-x)z$ is a regular sequence in $K[x, y, z]$ but $(1-x)y$, $(1-x)z$, x is not), but this is true in the local case and certain graded cases:

Proposition. *Let (R, m) be local and $M \neq 0$ finitely generated or let R be $\mathbb{N}$-graded, $M \neq 0$ $\mathbb{Z}$-graded with nonzero components bounded below, and let m be the ideal of R generated by all elements of positive degree. Let $x_1, \ldots, x_n$ be a sequence of elements of m (that are homogeneous of positive degree in the graded case). If the elements form a regular sequence on M in one order, they form a regular sequence on M in every order.*

Proof. Note that under these hypotheses $(x_1, \ldots, x_n)M \neq M$ is automatic, by Nakayama's lemma. }}medskip

Since the permutation group is generated by transpositions of consecutive elements, it suffices to show that given a regular sequence, the result of transposing consecutive elements x_i and x_{i+1} is a regular sequence. For the purpose, we may replace M by $M/(x_1, \ldots, x_{i-1})M$. Thus, we may assume the two elements are x_1 and x_2. The conditions that are imposed mod $(x_1, x_2)M$ or on the quotient $M/(x_1, \ldots, x_j)M$ for $j \geq 2$ are clearly not affected by the transposition. Therefore it suffices consider the case $n = 2$.

Assume that x_1, x_2 is a regular sequence on M. We first show that x_2 is not a zerodivisor. Let $N \subseteq M$ be the annihilator of x_2, which is again graded in the graded case. If $u \in N$, then $x_2 u = x_1(0)$, and so $u = x_1 v$ for some $v \in M$. Then $x_2 x_1 v = 0$ and since x_1 is not a zerodivisor on M, we have that $x_2 v = 0$ and $v \in N$. This shows that $N = x_1 N$. By Nakayama's lemma, $N = 0$, as required. We next show that x_1 is not a zerodivisor mod $x_2 M$. If $x_1 u = x_2 w$ then, since x_1, x_2 is a regular sequence we have that $w = x_1 v$ and $x_1(u - x_2 v) = 0$. Since x_1 is not a zerodivisor, $u = x_2 v$, as required. $\square$

We place special emphasis on the graded case here for several reasons. One is its importance in projective geometry. Beyond that, there are many theorems about the graded case that make it easier both to understand and to do calculations. Moreover, many of the most important examples of Cohen-Macaulay rings are graded.

We first note:

Proposition. *Let M be an $\mathbb{N}$-graded or $\mathbb{Z}$-graded module over an $\mathbb{N}$-graded or $\mathbb{Z}$-graded Noetherian ring S. Then every associated prime of M is homogeneous. Hence, every minimal prime of the support of M is homogeneous and, in particular the associated (hence, the minimal) primes of S are homogeneous.*

Proof. Any associated prime P of M is the annihilator of some element u of M, and then every nonzero multiple of $u \neq 0$ can be thought of as a nonzero element of $S/P \cong Su \subseteq M$, and so has annihilator P as well. If u_i is a nonzero homogeneous component of u of degree i, its annihilator J_i is easily seen to be a homogeneous ideal of S. If $J_h \neq J_i$ we can choose a form F in one and not the other, and then Fu is nonzero with fewer homgeneous components then u. Thus, the homogeneous ideals J_i are all equal to, say, J, and clearly $J \subseteq P$. Suppose that $s \in P - J$ and subtract off all components of S that are in J, so that no nonzero component is in J. Let $s_a \notin J$ be the lowest degree component of s and u_b be the lowest degree component in u. Then $s_a u_b$ is the only term of degree $a + b$ occurring in $su = 0$, and so must be 0. But then $s_a \in \mathrm{Ann}_S u_b = J_b = J$, a contradiction. $\square$

Corollary. *Let K be a field and let R be a finitely generated $\mathbb{N}$-graded K-algebra with $R_0 = K$. Let $\mathcal{M} = \bigoplus_{d=1}^{\infty} R_j$ be the homogeneous maximal ideal of R. Then* $\dim(R) = \text{height}(\mathcal{M}) = \dim(R_{\mathcal{M}})$.

Proof. The dimension of R will be equal to the dimension of R/P for one of the minimal primes P of R. Since P is minimal, it is an associated prime and therefore is homogenous.

Hence, $P \subseteq \mathcal{M}$. The domain R/P is finitely generated over K, and therefore its dimension is equal to the height of every maximal ideal including, in particular, $\mathcal{M}/P$. Thus,

$$\dim(R) = \dim(R/P) = \dim\big((R/P)_{\mathcal{M}}\big) \le \dim R_{\mathcal{M}} \le \dim(R),$$

and so equality holds throughout, as required. $\square$

Proposition (homogeneous prime avoidance). *Let R be an $\mathbb{N}$-graded algebra, and let I be a homogeneous ideal of R whose homogeneous elements have positive degree. Let $P_1, \ldots, P_k$ be prime ideals of R. Suppose that every homogeneous element $f \in I$ is in $\bigcup_{i=1}^k P_i$. Then $I \subseteq P_j$ for some j, $1 \le j \le k$.*

Proof. We have that the set H of homogeneous elements of I is contained in $\bigcup_{i=1}^k P_k$. If $k = 1$ we can conclude that $I \subseteq P_1$. We use induction on k. Without loss of generality, we may assume that H is not contained in the union of any $k-1$ if the P_j. Hence, for every i there is a homogeous element $g_i \in I$ that is not in any of the P_j for $j \ne i$, and so it must be in P_i. We shall show that if $k > 1$ we have a contradiction. By raising the g_i to suitable positive powers we may assume that they all have the same degree. Then $g_1^{k-1} + g_2 \cdots g_k \in I$ is a homogeneous element of I that is not in any of the P_j: g_1 is not in P_j for $j > 1$ but is in P_1, and $g_2 \cdots g_k$ is in each of $P_2, \ldots, P_k$ but is not in P_1. $\square$

Now suppose that R is a finitely generated $\mathbb{N}$-graded algebra over $R_0 = K$, where K is a field. By a *homogenous system of parameters* for R we mean a sequence of homogeneous elements $F_1, \ldots, F_n$ of positive degree in R such that $n = \dim(R)$ and $R/F_1, \ldots, F_n)$ has Krull dimension 0. When R is a such a graded ring, a homogeneous system of parameters always exists. By homogeneous prime avoidance, there is a form F_1 that is not in the union of the minimal primes of R. Then $\dim(R/F_1) = \dim(R) - 1$. For the inductive step, choose forms of positive degree $F_2, \ldots, F_n$ whose images in R/F_1R are a homogeneous system of parameters for R/F_1R. Then $F_1, \ldots, F_n$ is a homogeneous system of parameters for R. $\square$

Moreover, we have:

Theorem. *Let R be a finitely generated $\mathbb{N}$-graded K-algebra with $R_0 = K$ such that $\dim(R) = n$. A homogeneous system of parameters $F_1, \ldots, F_n$ for R always exists. Moreover, if $F_1, \ldots, F_n$ is a sequence of homogeneous elements of positive degree, then the following statements are equivalent.*

(1) *$F_1, \ldots, F_n$ is a homogeneous system of parameters.*
(2) *m is nilpotent modulo $(F_1, \ldots, F_n)R$.*
(3) *$R/(F_1, \ldots, F_n)R$ is finite-dimensional as a K-vector space.*
(4) *R is module-finite over the subring $K[F_1, \ldots, F_n]$.*

Moreover, when these conditions hold, $F_1, \ldots, F_n$ are algebraically independent over K, so that $K[F_1, \ldots, F_n]$ is a polynomial ring.

Proof. We have already shown existence.

(1) $\Rightarrow$ (2). If $F_1, \ldots, F_n$ is a homogeneous system of parameters, we have that

$$\dim\big(R/F_1, \ldots, F_n)\big) = 0.$$

We then know that all prime ideals are maximal. But we know as well that the maximal ideals are also minimal primes, and so must be homogeneous. Since there is only one homogenous maximal ideal, it must be $m/(F_1, \ldots, F_n)R$, and it follows that m is nilpotent on $(F_1, \ldots, F_n)R$.

(2) $\Rightarrow$ (3). If m is nilpotent modulo $(F_1, \ldots, F_n)R$, then the homogeneous maximal ideal of $\overline{R} = R/(F_1, \ldots, F_n)R$ is nilpotent, and it follows that $[\overline{R}]_d = 0$ for all $d \gg 0$. Since each $\overline{R}_d$ is a finite dimensional vector space over K, it follows that $\overline{R}$ itself is finite-dimensional as a K-vector space.

(3) $\Rightarrow$ (4). This is immediate from the homogeneous form of Nakayama's Lemma: a finite set of homogeneous elements of R whose images in $\overline{R}$ are a K-vector space basis will span R over $K[F_1, \ldots, F_n]$, since the homogenous maximal ideal of $K[F_1, \ldots, F_n]$ is generated by $F_1, \ldots, F_n$.

(4) $\Rightarrow$ (1). If R is module-finite over $K[F_1, \ldots, F_n]$, this is preserved mod $(F_1, \ldots, F_n)$, so that $R/(F_1, \ldots, F_n)$ is module-finite over K, and therefore zero-dimensional as a ring.

Finally, when R is a module-finite extension of $K[F_1, \ldots, F_n]$, the two rings have the same dimension. Since $K[F_1, \ldots, F_n]$ has dimension n, the elements $F_1, \ldots, F_n$ must be algebraically independent. $\square$

The technique described in the discussion that follows is very useful both in the local and graded cases.

Discussion: making a transition from one system of parameters to another. Let R be a Noetherian ring of Krull dimension n, and assume that one of the two situations described below holds.

(1) (R, m, K) is local and $f_1, \ldots, f_n$ and $g_1, \ldots, g_n$ are two systems of parameters.
(2) R is finitely generated $\mathbb{N}$-graded over $R_0 = K$, a field, m is the homogeneous maximal ideal, and $f_1, \ldots, f_n$ and $g_1, \ldots, g_n$ are two homogeneous systems of parameters for R.

We want to observe that in this situation there is a finite sequence of systems of parameters (respectively, homogeneous systems of parameters in case (2)) starting with $f_1, \ldots, f_n$ and ending with $g_1, \ldots, g_n$ such that any two consecutive elements of the sequence agree in all but one element (i.e., after reordering, only the i th terms are possibly different for a single value of i, $1 \le i \le n$). We can see this by induction on n. If $n = 1$ there is nothing to prove. If $n > 1$, first note that we can choose h (homogeneous of positive degree in the graded case) so as to avoid all minimal primes of $(f_2, \ldots, f_n)R$ and all minimal primes of $(g_2, \ldots, g_n)R$. Then it suffices to get a sequence from $h, f_2, \ldots, f_n$ to $h, g_2, \ldots, g_n$, since the former differs from $f_1, \ldots, f_n$ in only one term and the latter differs from $g_1, \ldots, g_n$ in only one term. But this problem can be solved by working in R/hR and getting a sequence from the images of $f_2, \ldots, f_n$ to the images of $g_2, \ldots, g_n$, which we can do by the induction hypothesis. We lift all of the systems of parameters back to R by taking, for each one, h and inverse images of the elements in the sequence in R (taking a homogeneous inverse image in the graded case), and always taking the same inverse image for each element of R/hR that occurs. $\square$

The following result now justifies several assertions about Cohen-Macaulay rings made without proof earlier.

Theorem. *Let (R, m, K) be a local ring. There exists a system of parameters that is a regular sequence if and only if every system of parameters is a regular sequence. In this case, for every prime P ideal I of R of height k, there is a regular sequence of length k in I.*

Moreover, for every prime ideal P of R, R_P also has the property that every system of parameters is a regular sequence.

Proof. For the first statement, we can choose a chain as in the comparison statement just above. Thus, we can reduce to the case where the two systems of parameters differ in only one element. Because systems of parameters are permutable and regular sequences are permutable in the local case, we may assume that the two systems agree except possibly for the last element. We may therefore kill the first $\dim(R) - 1$ elements, and so reduce to the case where x and y are one element systems of parameters in a local ring R of dimension 1. Then x has a power that is a multiple of y, say $x^h = uy$, and y has a power that is a multiple of x. If x is not a zerodivisor, neither is x^h, and it follows that y is not a zerodivisor. The converse is exactly similar.

Now suppose that I is any ideal of height h. Choose a maximal sequence of elements (it might be empty) of I that is part of a system of parameters, say $x_1, \ldots, x_k$. If $k < h$, then I cannot be contained in the union of the minimal primes of $(x_1, \ldots, x_k)$: otherwise, it will be contained in one of them, say Q, and the height of Q is bounded by k. Chose $x_{k+1} \in I$ not in any minimal prime of $(x_1, \ldots, x_k)R$. Then $x_1, \ldots, x_{k+1}$ is part of a system of parameters for R, contradicting the maximality of the sequence $x_1, \ldots, x_k$.

Finally, consider the case where $I = P$ is prime. Then P contains a regular sequence $x_1, \ldots, x_k$, which must also be regular in R_P, and, hence, part of a system of parameters. Since $\dim(R_P) = k$, it must be a system of parameters. $\square$

Lemma. *Let K be a field and assume either that*

(1) *R is a regular local ring of dimension n and $x_1, \ldots, x_n$ is a system of parameters*

or

(2) *$R = K[x_1, \ldots, x_n]$ is a graded polynomial ring over K in which each of the x_i is a form of positive degree.*

Let M be a nonzero finitely generated R-module which is $\mathbb{Z}$-graded in case (2). Then M is free if and only if $x_1, \ldots, x_n$ is a regular sequence on M.

Proof. The "only if" part is clear, since $x_1, \dots, x_n$ is a regular sequence on R and M is a direct sum of copies of R. Let $m = (x_1, \dots, x_n)R$. Then $V = M/mM$ is a finite-dimensional K-vector space that is graded in case (2). Choose a K-vector space basis for V consisting of homogeneous elements in case (2), and let $u_1, \dots, u_h \in M$ be elements of M that lift these basis elements and are homogeneous in case (2). Then the u_j span M by the relevant form of Nakayama's Lemma, and it suffices to prove that they have no nonzero relations over R. We use induction on n. The result is clear if $n = 0$.

Assume $n > 0$ and let $N = \{(r_1, \dots, r_h) \in R^h : r_1u_1 + \cdots + r_hu_h = 0\}$. By the induction hypothesis, the images of the u_j in M/x_1M are a free basis for M/x_1M. It follow that if $\rho = (r_1, \dots, r_h) \in N$, then every r_j is 0 in R/x_1R, i.e., that we can write $r_j = x_1s_j$ for all j. Then $x_1(s_1u_1 + \cdots + s_hu_h) = 0$, and since x_1 is not a zerodivisor on M, we have that $s_1u_1 + \cdots + s_hu_h = 0$, i.e., that $\sigma = (s_1, \dots, s_h) \in N$. Then $\rho = x_1\sigma \in x_1N$, which shows that $N = x_1N$. Thus, $N = 0$ by the appropriate form of Nakayama's Lemma. □

We next observe:

Theorem. *Let R be a finitely generated graded algebra of dimension n over $R_0 = K$, a field. Let m denote the homogeneous maximal ideal of R. The following conditions are equivalent.*

(1) *Some homogeneous system of parameters is a regular sequence.*
(2) *Every homogeneous system of parameters is a regular sequence.*
(3) *For some homogeneous system of parameters $F_1, \dots, F_n$, R is a free-module over $K[F_1, \dots, F_n]$.*
(4) *For every homogeneous system of parameters $F_1, \dots, F_n$, R is a free-module over $K[F_1, \dots, F_n]$.*
(5) *R_m is Cohen-Macaulay.*
(6) *R is Cohen-Macaulay.*

Proof. The proof of the equivalence of (1) and (2) is the same as for the local case, already given above.

The preceding Lemma yields the equivalence of (1) and (3), as well as the equivalence of (2) and (4). Thus, (1) through (4) are equivalent.

It is clear that (6) ⇒ (5). To see that (5) ⇒ (2) consider a homogeneous system of parameters in R. It generates an ideal whose radical is m, and so it is also a system of parameters for R_m. Thus, the sequence is a regular sequence in R_m. We claim that it is also a regular sequence in R. If not, x_{k+1} is contained in an associated prime of $(x_1, \dots, x_k)$ for some k, $0 \le k \le n-1$. Since the associated primes of a homogeneous ideal are homogeneous, this situation is preserved when we localize at m, which gives a contradiction.

To complete the proof, it will suffice to show that (1) ⇒ (6). Let $F_1, \dots, F_n$ be a homogeneous system of parameters for R. Then R is a free module over $A = K[F_1, \dots, F_n]$, a polynomial ring. Let Q be any maximal ideal of R and let P denote its contraction to A, which will be maximal. These both have height n. Then $A_P \to R_Q$ is faithfully flat.

Since A is regular, A_P is Cohen-Macaulay. Choose a system of parameters for A_P. These form a regular sequence in A_P, and, hence, in the faithfully flat extension R_Q. It follows that R_Q is Cohen-Macaulay. $\square$

From part (2) of the Lemma on p. 5 we also have:

Theorem. *Let R be a module-finite local extension of a regular local ring A. Then R is Cohen-Macaulay if and only if R is A-free.*

It it is not always the case that a local ring $(R,\, m,\, K)$ is module-finite over a regular local ring in this way. But it does happen frequently in the complete case. Notice that the property of being a regular sequence is preserved by completion, since the completion $\widehat{R}$ of a local ring is faithfully flat over R, and so is the property of being a system of parameters. Hence, R is Cohen-Macaulay if and only if $\widehat{R}$ is Cohen-Macaulay.

If R is complete and contains a field, then there is a coefficient field for R, i.e., a field $K \subseteq R$ that maps isomorphically onto the residue class field K of R. Then, if $x_1, \, \ldots \, , x_n$ is a system of parameters, R turns out to be module-finite over the formal power series ring $K[[x_1, \, \ldots \, , x_n]]$ in a natural way. Thus, in the complete equicharacteristic local case, we can always find a regular ring $A \subseteq R$ such that R is module-finite over A, and think of the Cohen-Macaulay property as in the Theorem above.

Theorem. *Let R be a Cohen-Macaulay ring. Then the polynomial ring and the formal power series ring in finitely many variables over R is Cohen-Macaulay.*

Proof. Obviously, it suffices to consider the case of adjoining one variable. In the case of $R[[x]]$, note that x is in every maximal ideal $\mathcal{M}$: if $x \notin \mathcal{M}$, then $1 = gx + v$ for $v \in \mathcal{M}$, and this contradicts the fact that $1 - gx$ is invertible (the inverse is $1 + gx + g^2x^2 + \cdots + g^tx^t + \cdots$). It follows that $\mathcal{M}$ has the form $xR[[x]] + mR[[x]]$ for a maximal ideal m of R. But if $vectfn$ is a system of parameters that is a regular sequence in R_m, then $x, f_1, \, \ldots \, , f_n$ is a system of parameters that is a regular sequence in $R[[x]]_{\mathcal{M}}$: note that $R[[x]]_{\mathcal{M}}/(x) \cong R_m$.

In the polynomial ring case, every maximal ideal $\mathcal{M}$ of $R[x]$ lies over a prime P of R, and we may replace R by R_P. Hence, we may assume that (R, m) is local and that $\mathcal{M}$ is maximal ideal of $R[x]$ lying over m. Let $f_1, \, \ldots \, , f_n$ be maximal regular sequence in R. This is also a regular sequence in $R[x]$ and in $R[x]_{\mathcal{M}}$. The issues are unaffected by killing this regular sequence. We may therefore assume that m is nilpotent, so that $\dim(R) = 0$, and the maximal ideals of $R[x]$ are generated mod m by monic polynomials g that are irreducible mod m. We may lift g to a monic polynomial G of $R[x]$ of the same degree, and G is not a zerodivisor. Since $R[x]_{\mathcal{M}}$ has dimension 1 and contains G, it follows that G is a system of parameters whose only element is a nonzerodivisor. $\square$

In an arbitrary Noetherian ring, we define the *height*, denoted $\text{height}\,(I)$, of a proper ideal I to be the least height of any minimal prime of I. By convention $\text{height}\,(R) = +\infty$. Note that a regular sequence in a proper ideal I can never have length greater than $\text{height}\,(I)$: if it did, we could localize at a minimal prime P of I whose height is $\text{height}\,(I)$, and the image of the regular sequence is part of a system of parameters in R_P.

Proposition. *Let R be a Noetherian ring. The following three conditions are equivalent.*
(1) *R is Cohen-Macaualy.*
(2) *Every proper ideal I of R contains a regular sequence of length equal to* height (I).
(3) *Every maximal regular sequence in a proper ideal I of R has length equal to* height (I).

Moreover, if R is Cohen-Macaulay, every associated prime P of an ideal $x_1, \dots, x_n$ generated by a regular sequence has height n: thus, ideals generated by regular sequences have no embedded primes.

Proof. We first prove the final statement. If P is an associated prime of $x_1, \dots, x_n$ this remains true when we localize at P. Then, in the Cohen-Macaulay ring R_P, the image of $x_1, \dots, x_n$ can be extended to a system of parameters that is a regular sequence. But $PR_P/(x_1, \dots, x_n)$ consists entirely of zerodivisors, so that it is impossible to extend the image of $x_1, \dots, x_n$ in R_P to a longer regular sequence. This means that $R_P/(x_1, \dots, x_n)$ has dimension 0, and so P has height n, as claimed.

Clearly, (3) $\Rightarrow$ (2) $\Rightarrow$ (1) (a regular sequence of length equal to height m, where m is maximal, remains a regular sequence in R_m.) It remains to prove (1) $\Rightarrow$ (3). Let a proper ideal I be given and let $x_1, \dots, x_n$ be a longest possible regular sequence in I. Then the image of I must consist entirely of zerodivisors in $R/(x_1, \dots, x_n)$, and so I must be contained in an associated prime P of $x_1, \dots, x_n$. By the result of the first paragraph, the height of P is n. Hence $n \geq \text{height}\,(P) \geq \text{height}\,(I)$. But $\text{height}\,(I) \leq n$ by the remark just before the statement of the Proposition. $\square$

Theorem. *Let (R, m) be a Cohen-Macaulay local ring of Krull dimension d. Then any saturated chain of primes joining m to a minimal prime of R has length d. In particular, the quotient of R by any minimal prime of R has dimension d.*

Proof. If R has dimension 0, this is clear. We next want to show that if $P \subseteq Q$ are distinct primes of R with P minimal and there is no prime strictly between P and Q then $\text{height}\,(Q) = 1$. For this purpose, we may replace R by R_Q. Thus, we want to show that if (R, m) is local Cohen-Macaulay and there is a minimal prime P such that R/P has dimension 1, then R has dimension 1. Otherwise let x, y be a regular sequence in R. Since P is a minimal prime of R, it is an associated prime of (0) in R, and we have an embedding $R/P \hookrightarrow R$. Let $J \subseteq R$ be the nonzero ideal $\text{Ann}_R P$. We claim that J/xJ injects into R/xR, i.e., that $J \cap xR = xJ$. For if $xr \in J$, then $xrP = 0$. Since x is not a zerodivisor, it follows that $rP = 0$ and $r \in J$, as required. But J/xJ is killed by P and x, and $x \notin P$. Since R/P has dimension one, x is a parameter, and so $P + xR$ is m-primary. This means that $J/xJ \subseteq R/xR$ is killed by a power of m, contradicting the fact that $y \in m$ is supposedly not a zerodivisor on R/xR.

We complete the proof induction on the dimension d of R.

Suppose that $P_0 \subseteq \cdots \subseteq P_d = m$ and $P'_0 \subseteq \cdots \subseteq P'_e = m$ are two saturated chains from minimal primes to m. Then P_1 and P'_1 are both height one primes, and we can choose a nonzerodivisor $x \in P_1$ and a nonzerodivisor $x' \in P'_1$. Then $y = xx' \in P_1 \cap P'_1$ is a nonzerodivisor, and so each of P_1, P'_1 is a minimal prime of yR (this uses that P_1 and P'_1

have height one). Working mod yR we get two saturated chains from a minimal prime of R/yR to the maximal ideal m/yR of the Cohen-Macaulay ring R/yR, one from P_1/yR to m/yR and one from P_1'/yR to m/yR. It follows from the induction hypothesis that $d-1 = e-1$ and so $d = e$. $\square$

Corollary. *Every Cohen-Macaulay ring is universally catenary.*

Proof. A finitely generated algebra over a Cohen-Macaulay ring R is a homomorphic image over a polynomial ring S in finitely many variables over R. Since S is Cohen-Macaulay and the catenary property passes to quotient rings, it suffices to show that all saturated chains from P to Q in S have the same length. For this purpose we may replace S by S_Q. Furthermore, we may kill the ideal generated by a maximal regular sequence $x_1, \ldots, x_h$ in PS_Q. We have now reduced to the case of a local Cohen-Macaulay ring in which P is minimal and Q is the maximal ideal. $\square$

Examples. Every zero-dimensional Noetherian ring is Cohen-Macaulay. Every one-dimensional reduced Noetherian ring is also Cohen-Macaulay, since a local reduced ring of dimension one every element not in any minimal prime is a nonzerodivisor. In particular, every one-dimensional Noetherian domain is Cohen-Macaulay.

Throughout the examples that follow, K is a field. Let x, y be indeterminates over K. $K[x,y]/(y^2)$ is Cohen-Macaulay (it is a quotient of a regular ring by an ideal generated by a regular sequence). So the condition that the one-dimensional ring be reduced is sufficient but not necessary for the Cohen-Macaulay property. The ring $K[x,y]/(x^2, xy)$ is not Cohen-Macaulay even when localized at at (x,y). The image of x is nonzero but kills the maximal ideal, and so every element of the maximal ideal is a zerodivisor.

A normal ring of dimension at most two is Cohen-Macaulay. The localization at any height one prime is a DVR. If one localizes at a height two prime and x, y is a system of parameters, then every associated prime of xR has height one, and so y is not in any of the associated primes of xR, which implies that y is not a zero divisor on R/xR. On the other hand $K[x,y,z]/(x^2)$ is a two-dimensional Cohen-Macaulay ring that is not reduced.

The two-dimensional non-normal rings $K[x^2, x^3, y, xy]$ and $K[x^4, x^3y, xy^3, y^4]$, both subring of the polynomial ring $K[x,y]$ are not Cohen-Macaulay. In the first, x^2, y is a homogeneous system of parameters but not a regular sequence since $x^3(y) = (xy)x^2$, but x^3 is not a multiple of x^2 in this ring. In the second, x^4, y^4 is a homogeneous system of parameters but $(x^3y)^2y^4 = (xy^3)^2x^4$ while $(x^3y)^2$ is not a multiple of x^4 in this ring.

Let X, Y, Z, s, t be indeterminates over a field K of characteristic not 3. If $K[x,y,z] = K[X,Y,Z]/(X^3+Y^3+Z^3)$, and $R = K[xs, ys, zs, xt, yt, zt] \subseteq K[x,y,z,s,t] = S$. Note that S is Cohen-Macaulay, and it turns out that S is normal as well. S is graded by $\mathbb{N} \times \mathbb{N}$ such that x, y, z have degree $(1,0)$ and s, t have degree $(0,1)$. $R \to S$ splits as a map of R-modules: the splitting kills the span of the homogeneous elements of degree (d, d') for $d \neq d'$ and is the identity on the homogeneous elements of degree (d,d), (the latter span R over K). It follows that R is normal. Also, R is module-finite over $K[xs, ys, xt, yt]$ and has a homogeneous system of parameters consisting of xs, $yt, xt - ys$. R is not Cohen-Macaulay because one has $(zs)(zt)(xt - ys) = (zt)^2xs - (zs)^2yt$, but $(zs)(zt) \notin (ys, xt)$ even in $K[x, y, z, s, t]$ after specializing $s \mapsto 1$, $t \mapsto 1$, since $z^2 \notin (x, y)$ in $K[x, y, z]$.

As mentioned earlier, the quotient of a Cohen-Macaulay ring by the ideal generated by a regular sequin is Cohen-Macaulay, and, in particular the quotient of a regular ring by an ideal generated by a regular sequence is Cohen-Macaulay.

There is a considerable literature studying whether specific rings are Cohen-Macaulay. The ring generated by the $n \times n$ minors of an $n \times s$ matrix of indeterminates over K is Cohen-Macaulay (cf. [M. Hochster, *Grassmannians and their Schubert subvarieties are arithmetically Cohen-Macaulay*, J. of Algebra **25** (1973), 40–57]), the ring obtained by killing the ideal generated by the size t minors of an $r \times s$ matrix of indeterminates is Cohen-Macaulay (cf. [J. A. Eagon and M. Hochster, *Cohen-Macaulay rings, invariant theory, and the generic perfection of determinantal loci*, Amer. J. Math. **93** (1971), 1020–1058]), and any normal ring generated by monomials of a polynomial ring over a field is Cohen-Macaulay(cf. [M. Hochster, *Rings of invariants of tori, Cohen-Macaulay rings generated by monomials, and polytopes*, Annals of Math. **96** (1972), 318–337])