Ideals of Minors and Fitting Invariants

Let A denote an $s \times h$ matrix over a ring R. Let t be a nonnegative integer. We define $I_t(A)$ to be the ideal generated by the $t \times t$ minors of A if $1 \leq t \leq \min\{s, h\}$. If $t = 0$, we may the convention $I_0(A) = R$. If $t > \min\{s, h\}$ we make the convention that $I_t(A) = 0$. If B is an $r \times s$ matrix, then $I_t(BA) \subseteq I_t(B)I_t(A) \subseteq I_t(A)$. The reason is that A represents an R-linear map $R^h \to R^s$, and $\bigwedge^t A : \bigwedge^t(R^s) \to \bigwedge^t(R^r)$ has a matrix whose entries are the $t \times t$ minors of A. Since $\bigwedge^t(BA) = \bigwedge^t(B) \bigwedge^t(A)$, it follows that each size t minor of BA is in $I_t(A)I_t(B)$. In particular, $I_t(BA) \subseteq I_t(A)$ and $I_t(BA) \subseteq I_t(B)$. Hence, if B is invertible, $I_t(BA) = I_t(A)$, since $I_t(BA) \subseteq I_t(A)$ and $I_t(A) = I_t\big(B^{-1}(BA)\big) \subseteq I_t(BA)$. Similarly, if C is an $s \times s$ invertible matrix, $I_t(AC) = I_t(A)$. Hence, performing elementary row operations on a matrix A (adding a multiple of one row to another, permuting the rows, multiplying a row by a unit) does not affect $I_t(A)$, and the same holds for elementary column operations.

For any finitely presented module M over any ring R, we can define *Fitting invariants*: the i th Fitting invariant of M is the ideal $I_{s-i}(J)$ where J is the $s \times h$ matrix of the map of free modules in a presentation $R^h \to R^s \twoheadrightarrow M \to 0$.

Fitting's Lemma. *The i th Fitting invariant as defined in the preceding paragraph is independent of the choice of finite presentation of M.*

Proof. To prove this, we first check that given a map $R^s \twoheadrightarrow M$, the i th Fitting invariant is independent of the choice of the finitely many column vectors spanning the kernel. Given two choices, we may compare each with the union. This, it suffices to see that the ideal does not change when one set of relations is included in the other. We may think of one matrix as $s \times h$ and we may assume that it is the submatrix of the other, which is $s \times (h+k)$, formed from the first h columns, while the last k columns are linear combinations of the first h columns. By subtracting linear combinations of the first h columns from the last k (we know that this does not change the ideals of minors) we may assume that the last k columns are all 0, and the result is now clear.

It remains to check independence of the map $R^s \twoheadrightarrow M$, i.e., of the choice of generators for M. Again, we may compare each of two different sets of generators with their union, and so we reduce to the case where one set of generators is included in the other and then, by induction, to the case where there is one additional generator. We may assume that included among the relations is a relation expressing the additional generator, which we number last, as a linear combination of the others. This means that we may assume that the matrix with the additional generators present has a 1 in the last row, which we also assume, by permuting the columns, is in the last column. We can now perform elementary column operations, subtracting multiples of the last column from the others, until the last row consists of all zeros except for its final entry, so that the $(s+1) \times (h+1)$ matrix J_1

that we are considering has the block form $A = \begin{pmatrix} B & C \\ 0 & 1 \end{pmatrix}$, where B is $s \times h$, 0 denotes a row of zeros of length h, C is an $s \times 1$ column, and 1 is the 1×1 identity matrix. Then B gives a matrix for the presentation using the first s generators. It is now straightforward to see that $I_{s+1-i}(A) = I_{s-i}(B)$. Let $t = s - i$. Each $t+1$ size minor of A that involves the element in the lower right corner is the same, up to sign, as a t size minor of B, all of which occur. We need to check that the other $t+1$ size minors of A are in $I_t(B)$. If the minor involves the last row, it is 0. Otherwise, it has at least t columns in B, and its expansion by minors with respect to the remaining column is therefore in $I_t(B)$. $\square$