Module-finite Extensions of Complete Local Rings

Theorem. *A module-finite extension S of a complete local ring (R, m, K) is a finite product of complete local rings: moreover, there is a bijection between the factors and the maximal ideals of S. Therefore, if S is a domain (then R will automatically be a domain as well), S is complete and local.*

Proof. Let Q be any maximal ideal of S, and suppose it contracts to P in R. Then $R/P \hookrightarrow S/Q$ is module-finite, and so R/P has Krull dimension 0. Therefore, Q lies over m. The primes of S lying over m correspond to the primes of S/mS, which is module-finite over $R/m = K$ and so zero-dimensional. Therefore, S has only finitely many maximal ideals, all of which lie over m. Let $Q_1, \dots, Q_n$ be the maximal ideals of S. Let $J = \bigcap_{i=1}^n Q_n$. Then $\mathrm{Rad}\,(mS) = J$, and J has a power in mS, say J^k. Since S is module-finite over R and R is complete, S is m-adically complete. Since $J^k \subseteq mS \subseteq J$, S is J-adically complete. Note that $J^t = (Q_1 \cap \cdots \cap Q_n)^t = (Q_1 \cdots Q_n)^t$ (since the Q_i are pairwise comaximal) $= Q_1^t \cap \cdots \cap Q_n^t = Q_1^t \cap \cdots \cap Q_n^t$ (since the Q_i^t are pairwise comaximal). Hence,

$$S \cong \varprojlim_t S/J^t \cong \varprojlim_t S/(Q_1^t \cap \cdots \cap Q_n^t) = \varprojlim_t \big((S/Q_1^t) \times \cdots \times (S/Q_n^t)\big)$$

by the Chinese Remainder theorem. Since limit (or inverse limit) commutes with finite products, this is

$$(\varprojlim_t S/Q_1^t) \times \cdots \times (\varprojlim_t S/Q_n^t).$$

When Q is maximal in S, S/Q^t is already a local ring (the only prime that contains Q^t is Q) and so $S/Q^t \cong S_Q/(QS_Q)^t$. Hence, $\varprojlim_t S/Q_i^t$ may be identified with $(Q_iS_{Q_i})-$adic completion of S_{Q_i}. Thus, S is isomorphic with a product of complete local rings, one for each of its maximal ideals. Evidently, S cannot be a domain unless there is only one factor, in which case S is local. $\square$