Regular Rings, Finite Projective Resolutions, and Grothendieck Groups

Recall that a local ring $(R,\, m,\, K)$ is *regular* if its embedding dimension, $\dim_K(m/m^2)$, which may also be described as the least number of generators of the maximal ideal m, is equal to its Krull dimension. This means that a minimal set of generators of m is also a system of parameters. Such a system of parameters is called *regular*. Another equivalent condition is the the associated graded ring of R with respect to m be a polynomial ring, in which case the number of variables is the same as $\dim(R)$. (The associated graded ring will be generated minimally by m/m^2, and has the same Krull dimension as R. It follows at once that if it is polynomial, then R is regular. But if R is regular, there cannot be any algebraic relations over K on a basis for m/m^2, or the Krull dimension of $\mathrm{gr}_m R$ will be smaller than that of R.

It is easy to see that a regular local ring is a domain: in fact, whenever $\mathrm{gr}_m R$ is a domain, R is a domain (if $a \in m^h - m^{h+1}$ and $b \in m^k - m^{k+1}$ are nonzero, then ab cannot be 0, or even in m^{h+k+1}, or else the product of the images of a in m^h/m^{h+1} and m^k in m^k/m^{k+1} will be 0 in $\mathrm{gr}_m R$. We shall see eventually that regular local rings are UFDs and, in particular, are normal.

The following fact about regular local rings comes up frequently.

Proposition. *Let $(R,\, m,\, K)$ be a regular local ring. Let $J \subseteq m$ be a proper ideal of R. Then R/J is regular if and only if J is generated by part of a minimal set of generators for m, i.e., part of a regular system of parameters. (This is true if $J = 0$, since we may take the set to be empty.)*

Proof. If J is generated by $x_1,\, \ldots,\, x_k$, part of a minimal set of generators for m, then x_1 is not in an minimal prime, since R is a domain, and both the dimension and the embedding dimension of R/x_1R are one less than the corresponding number for R. It follows that R/x_1R is again regular, and the full result follows by a straightforward induction on k.

To prove the other direction, we also use induction on $\dim(R)$. The case where $\dim(R) = 0$ is obvious. Suppose $\dim(R) > 0$ and $0 \neq J \subseteq m^2$. Then R/J is not regular, for its dimension is strictly less than that of R, but its embedding dimension is the same. Thus, we may assume instead that there exists an element $x_1 \in J$ with $x_1 \notin m^2$, so that x_1 is part of a minimal set of generators for m. Then R/x_1R is again regular, and $(R/x_1R)/(J/x_1R) \cong R/J$ is regular. It follows that J/x_1R is generated by part of a minimal system of generators $\overline{x}_2,\, \ldots,\, \overline{x}_k$ for m/x_1R, where $\overline{x}_j$ is the image in R/x_1R of $x_j \in m$, $2 \leq j \leq k$. But then $x_1,\, \ldots,\, x_k$ is part of a minimal set of generators for m. $\square$

Projective resolutions

We want to characterize regular local rings in terms of the existence of finite free resolutions. Note that over a local ring, a finitely generated module is flat iff it is projective iff it s free.

Over any ring R, every module has a projective resolution. That is, given M, there is a (usually infinite) exact sequence $\cdots \to G_n \to \cdots \to G_1 \to G_0 \to M \to 0$, such that all of the G_i are projective. In fact, we may take them all to be free.

One can construct a free resolution as follows. First choose a set of generators $\{u_\lambda\}_{\lambda\in\Lambda}$ for M, and then map the free module $G_0 = \bigoplus_{\lambda\in\Lambda} Rb_\lambda$ on a correspondingly indexed set of generators $\{b_\lambda\}_{\lambda\in\Lambda}$ onto M: there is a unique R-linear map $G_0 \twoheadrightarrow M$ that sends $b_\lambda \mapsto u_\lambda$ for all $\lambda \in \Lambda$. Whenever we have such a surjection, the kernel M_1 of $P \twoheadrightarrow M$ is referred to as a *first module of syzygies* of M. If $0 \to M_1 \to P_0 \to M \to 0$ is exact with P_0 free, we may repeat the process and form an exact sequence $0 \to M_2 \to P_1 \to M_1 \to 0$. Then the sequence $0 \to M_2 \to P_1 \to P_0 \to M \to 0$ is also exact, where the map $P_1 \to P_0$ is the composition of the maps $P_1 \twoheadrightarrow M_1$ and $M_1 \hookrightarrow P_0$.

Recursively, we may form short exact sequences $0 \to M_n \to P_{n-1} \to M_{n-1} \to 0$ for all $n \geq 1$ (where $M_0 = M$), and then one has that every $n \geq 1$, the sequence

$$(*)\quad 0 \to M_n \to P_{n-1} \to P_{n-2} \to \cdots \to P_3 \to P_2 \to P_1 \to P_0 \to M \to 0$$

is exact. A module M_n that occurs in such an exacct sequence $(*)$ in which all the P_i are projective modules is called an n th module of syzygies of M. Equvialently, an n th module of syzygies may be defined recursively as a first module of syzygies of any $n-1$ st module of syzygies. Note that the (usually infinite) sequence

$$(**)\quad \cdots \to P_n \to P_{n-1} \to \cdots \to P_3 \to P_2 \to P_1 \to P_0 \to M \to 0$$

is exact as well, and so is a projective resolution of M.

A projective resolution is called *finite* if $P_n = 0$ for all $n \gg 0$. If M has a finite projective resolution, it is said to have *finite projective dimension*. The projective dimension of M is defined to be -1 if $M = 0$ and to be 0 if M is nonzero and projective. In general, if M has finite projective dimension and is not projective, the projective dimension n of M, which we denote $\mathrm{pd}_R M$ (or simply $\mathrm{pd}\, M$ if R is understood from context), is the smallest integer n for which one can find a finite projective resolution $0 \to P_n \to \cdots \to P_1 \to P_0 \to M \to 0$. If M does not have finite projective dimension, it is said to have infinite projective dimension, and one may write $\mathrm{pd}\, M = \infty$.

If R is Noetherian and M is finitely generated, one may also construct a module of syzygies by mapping a finitely generated free module onto M. The first module of syzygies will then be a submodule of this finitely generated free module, and, hence, finitely generated again. Therefore, M has a free resolution by finitely generated free modules.

In the sequel, we prove the following result:

Theorem (Auslander-Buchsbaum-Serre. *Let (R, m, K) be a local ring. Then the following conditions are equivalent.*
(1) *R is regular.*
(2) *The residue field $K = R/m$ has a finite free resolution.*
(3) *Every finitely generated R-module has a finite free resolution.*

Before giving a proof, which will be based on elementary properties of Tor, we note an important consequence of this characterization of regularity.

Corollary. *If R is a regular local ring and Q is a prime ideal of R, then R_Q is regular.*

Proof. Since R is regular, R/Q has a finite R-free resolution by R-modules. We may then localize at Q to obtain a finite R_Q-free resolution of the residue class field $R_Q/QR_Q \cong (R/Q)_Q$. $\square$

Thus, a Noetherian ring has the property that its localization at every prime ideal is regular if and only if it has the property that its localization at every maximal ideal is regular. A Noetherian ring with these equivalent properties is called *regular.*

Minimal free resolutions over local rings

Let (R, m, K) be local. We keep the notation of the preceding section. In constructing a free resolution for a finitely generated R-module M, we may begin by choosing a minimal set of generators for M. Then, at every stage, we may choose a minimal set of generators of M_{n-1} and use that minimal set to map a free R-module onto M_{n-1}. A resolution constructed in this way is called a *minimal* free resolution of M. Thus, a free resolution $\cdots \to P_n \to \cdots \to P_1 \to P_0 \to M \to 0$ iof M is *minimal* precisely if every P_n that occurs has a free basis that maps to a minimal set set of generators of the image M_n of P_n.

Our discussion shows that minimal free resolutions exist. We also note the following fact: a free resolution $\to P_n \to P_{n-1} \to \cdots P_1 \to P_0 \to M \to 0$ (where the P_j are finitely generated free R-modules) is minimal if and only if for all $n \geq 1$, the image of P_j in P_{j_1} is contained in mP_{j-1}. The reason for this is that image of P_j consists of elements of P_{j-1} that give generators for the relations on the generators of M_{j-1}. These generators will be minimal generators if and only if they have no relation with a coefficient that is a unit, i.e., all of the generating relations are in mP_{j-1}. An equivalent way to phrase this is that entries of matrices for the maps $P_j \to P_{j-1}$ have all of their entries in m.

Recall that $\mathrm{Tor}_n(M, N)$ may be defined as the homology module at the nth spot of the complex $P_\bullet \otimes_R N$, where $P_\bullet$ is the complex $\cdots \to P_n \to \cdots \to P_0 \to 0$ obtained by replacing M by 0 in a projective resolutons $\cdots \to P_n \to \cdots \to P_0 \to M \to 0$. It is is independent of the specific projective resolution chosen up to canonical isomorphism. We assume familiarity with a few basic properties of Tor over a ring R as described in the supplement entitled *The Functor Tor.* The specific facts that we need about Tor^R (we frequently omit the superscript) are these:

(1) If $\cdots \to P_n \to \cdots \to P_0 \to M$ is a projective resolution of M, then $\mathrm{Tor}_n(M, N)$ is the homology of the complex $\cdots \to P_n \otimes N \to \cdots \to P_0 \otimes N \to 0$ at the $P_n \otimes N$ spot. Hence, $\mathrm{Tor}_n(M, N) = 0$ if $n > \mathrm{pd}\, M$.
(2) $\mathrm{Tor}_n(M, N) = 0$ for $n < 0$, and $\mathrm{Tor}_0(M, N) \cong M \otimes N$.
(3) $\mathrm{Tor}_n(M, N) \cong \mathrm{Tor}_n(N, M)$.
(4) $Tor_n(M, \_)$ (respectively, $\mathrm{Tor}_n(\_, M)$) is a covariant functor from R-modules to R-modules
(5) If $0 \to M' \to M \to M'' \to 0$ is exact there is a long exact sequence:
$\cdots \to \mathrm{Tor}_n(M', N) \to \mathrm{Tor}_n(M, N) \to \mathrm{Tor}_n(M'', N) \to \mathrm{Tor}_{n-1}(M', N) \to \cdots$
(6) The map induced on $\mathrm{Tor}_n(M, N)$ by multiplication by $r \in R$ on M (or N) is multiplication by r.
(7) The module $\mathrm{Tor}_n(M, N)$ is killed by $\mathrm{Ann}_R M + \mathrm{Ann}_R N$.
(8) If M or N is flat (e.g., if either is free or projective), then $\mathrm{Tor}_n(M, N) = 0$ for $n \geq 1$
(9) If R is Noetherian and M, N, are finitely generated, so is $\mathrm{Tor}_n(M, N)$ for all n.

From our discussion of minimal resolutions we obtain:

Theorem. *Let M be a finitely generated module over a local ring (R, m, K). The modules $Tor_i(M, K)$ are finite-dimensional vector spaces over K, and $\dim_K\big(Tor_i(M, K)\big)$ is the same as the rank of the i th free module in a minimal free resolution of M.*

Moreover the following conditions on M are equivalent:
(1) *In a minimal free resolution $P_\bullet$ of M, $P_{n+1} = 0$.*
(2) *The projective dimension of M is at most n.*
(3) *$Tor_{n+1}(M, K) = 0$.*
(4) *$Tor_i(M, K) = 0$ for all $i \geq n + 1$.*

It follows that a minimal free resolution of M is also a shortest possible projective resolution of M. In particular, M has finite projective dimension (respectively, infinite projective dimension) if and only its minimal free resolution is finite (respectively, infinite.)

Proof. If we take a minimal free resolution $P_\bullet$ of M, because the image of every G_j is in mG_{j-1}, when we apply $\_ \otimes_R K$ the maps become 0, while $G_i \otimes K$ is a vector space V_i over K whose dimension is the same as the rank of G_i. Hence, the homology of $P_\bullet \otimes_R K$ at the i th spot is V_i, and the first statement follows. It is clear that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1) (note that once one of the P_j is 0, all the P_k for $k \geq j$ are 0). The last implication follows from the first assertion of the Theorem. It is also clear that (1) $\Rightarrow$ (4) $\Rightarrow$ (3) $\Rightarrow$ (1). There cannot be a projective resolution shorter than the minimal resolution, for if $P_j \neq 0$ then $\mathrm{Tor}_j(M, K) \neq 0$, and if there were a shorter resolution it could be used to compute $\mathrm{Tor}_j(M, K)$, which would have to vanish. The final statement is then clear. $\square$

We shall next use some elementary facts about Tor to prove that over a regular local ring, every finitely generated module has finite projective dimension. We first note:

Proposition. *Let R be a ring and let $x \in R$ an element.*
(a) *Given an exact sequence $\mathcal{Q}_\bullet$ of modules*

$$\cdots \to Q_{n+1} \to Q_n \to Q_{n-1} \to \cdots$$

(it may be doubly infinite) such that x is a nonzerodivisor on all of the modules Q_n, the complex $\overline{\mathcal{Q}}_\bullet$ obtained by applying $\_ \otimes R/xR$, which we may alternatively describe as

$$\cdots \to Q_{n+1}/xQ_{n+1} \to Q_n/xQ_n \to Q_{n-1}/xQ_{n-1} \to \cdots ,$$

is also exact.

(b) *If x is a nonzerodivisor in R and is also a nonzerodivisor on the module M, while $xN = 0$, then for all i, $Tor_i^R(M, N) \cong Tor_i^{R/xR}(M/xM, N)$.*

Proof. (a) We get a short exact sequence of complexes $0 \to \mathcal{Q}_\bullet \xrightarrow{x\cdot} \mathcal{Q}_\bullet \to \overline{\mathcal{Q}}_\bullet \to 0$ which, at the n th spots, is $0 \to Q_n \xrightarrow{x\cdot} Q_n \to Q_n/xQ_n \to 0$ (exactness follows because x is a nonzerodivisor on every Q_n). The snake lemma yields that

$$\cdots \to H_n(\mathcal{Q}_\bullet) \to H_n(\mathcal{Q}_\bullet) \to H_n(\overline{\mathcal{Q}}) \to H_{n-1}(\mathcal{Q}_\bullet) \to \cdots$$

is exact, and since $H_n(\mathcal{Q}_\bullet)$ and $H_{n-1}(\mathcal{Q}_\bullet)$ both vanish, so does $H_n(\overline{\mathcal{Q}}_\bullet)$.

(b) Consider a free resolution $(*)$ $\cdots \to P_n \to \cdots P_0 \to M \to 0$ for M. By part (a), this remains exact when we apply $\_ \otimes_R R/xR$, which yields a free resolution of M/xM over R/xR. Let $P_\bullet$ be the complex $(*)$ with M replaced by 0. Then $\mathrm{Tor}_n^R(M, N)$ is the homology at the n th spot of $P_\bullet \otimes_R N$. Since x kills N, $(R/xR) \otimes_{R/xR} N \cong N$. Thus, $\mathrm{Tor}_n(M, N)$ is the homology at the n th spot of $\big(P_\bullet \otimes_R (R/xR)\big) \otimes_{R/xR} N$, and since $P_\bullet \otimes_R R/xR$ is a free resolution of M/xM over R/xR, this is also $\mathrm{Tor}_n^{R/xR}(M, N)$. $\square$

We can now prove:

Theorem. *If (R, m, K) is a regular local ring of Krull dimension d, then for every finitely generated R-module M, the projective dimension of M is at most d.*

Proof. We use induction on $\dim(R)$. If $\dim(R) = 0$, then the maximal ideal of R is generated by 0 elements, and is a field, so that every R-module is free and has projective dimension at most 0.

Now suppose $\dim(R) \geq 1$. Let M be a finitely generated R-module. It suffices to prove that $\mathrm{Tor}_n(M, K) = 0$ for $n > d$. We can form a short exact sequence $0 \to M_1 \to P \to M \to 0$ where P is free. Since $M_1 \subseteq P$, if we choose a regular paramter $x \in M$, x is not a zerodivisor on M_1. Hence, $\mathrm{Tor}_n^R(M_1, K) \cong \mathrm{Tor}_n^{R/xR}(M_1/xM_1, K)$ by the preceding Proposition. The long exact sequence for Tor coming from the short exact sequence $0 \to M_1 \to P \to M \to 0$ shows that $\mathrm{Tor}_{n+1}^R(M, K) \cong \mathrm{Tor}_n^R(M_1, K) \cong \mathrm{Tor}_n^{R/xR}(M_1/xM_1, K)$ for $n \geq d$, and the last term vanishes by the induction hypothesis, since R/xR is again regular. $\square$

The converse is much more difficult. We need several preliminary results. We write $\mathrm{pd}_R M$ or, if R is understood from context, $\mathrm{pd}\, M$ for the projective dimension of M over R. We first note:

Theorem. *Let $(R,\, m,\, K)$ be a local ring.*

Given a finite exact sequence of finitely generated R-modules such that every term but one has finite projective dimension, then every term has finite projective dimension.

In particular, given a short exact sequence

$$0 \to M_2 \to M_1 \to M_0 \to 0$$

of finitely generated R-modules, if any two have finite projective dimension over R, so does the third. Moreover:

(a) $pd\, M_1 \le \max\{pd\, M_0,\, pd\, M_2\}$.
(b) *If* $pd\, M_1 < pd\, M_0$ *are finite, then* $pd\, M_2 = pd\, M_0 - 1$*. If* $pd\, M_1 \ge pd\, M_0$*, then* $pd\, M_2 \le pd\, M_1$.
(c) $pd\, M_0 \le \max\{pd\, M_1, pd\, M_2 + 1\}$.

Proof. Consider the long exact sequence for Tor:

$$\cdots \to \mathrm{Tor}^R_{n+1}(M_1, K) \to \mathrm{Tor}^R_{n+1}(M_0, K) \to \mathrm{Tor}^R_n(M_2, K) \\ \to \mathrm{Tor}^R_n(M_1, K) \to \mathrm{Tor}^R_n(M_0, K) \to \cdots$$

If two of the M_i have finite projective dimension, then two of any three consecutive terms are eventually 0, and this forces the third term to be 0 as well.

The statements in (a), (b), and (c) bounding some $\mathrm{pd}\, M_j$ above for a certain $j \in \{0, 1, 2\}$ all follow by looking at trios of consecutive terms of the long exact sequence such that the middle term is $\mathrm{Tor}^R_n(M_j, K)$. For n larger than the specified upper bound for $\mathrm{pd}_R M_j$, the Tor on either side vanishes. The equality in (b) for the case where $\mathrm{pd}\, M_1 < \mathrm{pd}\, M_0$ follows because with $n = \mathrm{pd}\, M_0 - 1$, $\mathrm{Tor}^R_{n+1}(M_0,\, K)$ injects into $\mathrm{Tor}^R_n(M_2,\, K)$.

The statement about finite exact sequences of arbitrary length now follows by induction on the length. If the length is smaller than three we can still think of it as 3 by using terms that are 0. The case of length three has already been handled. For sequences of length 4 or more, say

$$0 \to M_k \to M_{k-1} \to \cdots \to M_1 \to M_0 \to 0,$$

either M_k and M_{k-1} have finite projective dimension, or M_1 and M_0 do. In the former case we break the sequence up into two sequences

$$0 \to M_k \to M_{k-1} \to B \to 0$$

and

$$(*) \quad 0 \to B \to M_{k-2} \to \cdots \to M_1 \to M_0 \to 0.$$

The short exact sequence shows that $\mathrm{pd}\, B$ is finite, and then we may apply the induction hypothesis to $(*)$. If M_1 and M_0 have finite projective dimension we use exact sequences

$$0 \to Z \to M_1 \to M_0 \to 0$$

and

$$0 \to M_k \to M_{k-1} \to \cdots \to M_2 \to Z \to 0$$

instead. $\square$

Lemma. *If M has finite projective dimension over (R, m, K) local, and $m \in \mathrm{Ass}\,(R)$, then M is free.*

Proof. If not, choose a minimal free resolution of M of length $n \geq 1$ and suppose that the left hand end is

$$0 \to R^b \xrightarrow{A} R^a \to \cdots$$

where A is an $a \times b$ matrix with entries in m. The key point is that the matrix A cannot give an injective map, because if $u \in m - \{0\}$ is such that $\mathrm{Ann}_R u = m$, then A kills a column vector whose only nonzero entry is u. $\square$

Lemma. *If M has finite projective dimension over R, and x is not a zerodivisor on R and not a zerodivisor on M, then M/xM has finite projective dimension over both R and over R/xR.*

Proof. Let $P_\bullet$ be a finite projective resolution of M over R. Then $P_\bullet \otimes_R R/xR$ is a finite complex of projective R/xR-modules whose homology is $\mathrm{Tor}_n^R(M,\, R/xR)$, which is 0 for $n \geq 1$ when x is not a zerodivisor on R or M. This gives an (R/xR)-projective resolution of M over R/xR. The short exact sequence

$$0 \to P \xrightarrow{x} P \to P/xP \to 0$$

shows that each P/xP has projective dimension at most 1 over R, and then M/xM has finite projective dimension over R by the Proposition above. $\square$

Lemma. *Let (R, m, K) be local, let I_n denote the $n \times n$ identity matrix over R, let x be an element of $m - m^2$, and let A, B be $n \times n$ matrices over R such that $xI_n = AB$. Suppose that every entry of A is in m. Then B is invertible.*

Proof. We use induction on n. If $n = 1$, we have that $(x) = (a)(b) = (ab)$, where $a \in m$. Since $x \notin m^2$, we must have that b is a unit. Now suppose that $n > 1$. If every entry of B is in m, the fact that $xI_n = AB$ implies that $x \in m^2$ again. Thus, some entry of B is a unit. We permute rows and columns of B to place this unit in the upper left hand corner. We multiply the first row of B by its inverse to get a 1 in the upper left hand corner. We next subtract multiples of the first column from the other columns, so that the first row becomes a 1 followed by a string of zeros. We then subtract multiples of the first row from the other rows, so that the first column becomes 1 with a column of zeros below it. Each of these operations has the effect of multiplying on the left or on the right by an invertible $n \times n$ matrix. Thus, we can choose invertible $n \times n$ matrices U and V over R such that $B' = UBV$ has the block form

$$B' = \begin{pmatrix} 1 & 0 \\ 0 & B_0 \end{pmatrix},$$

where the submatrices 1, 0 in in the first row are 1×1 and $1 \times (n-1)$, respectively, while the submatrices 0, B_0 in the second row are $(n-1) \times 1$ and $(n-1) \times (n-1)$, respectively.

Now, with

$$A' = V^{-1} A U^{-1},$$

we have

$$A'B' = V^{-1}AU^{-1}UBV = V^{-1}(AB)V = V^{-1}(xI_n)V = x(V^{-1}I_nV) = xI_n,$$

so that our hypothesis is preserved: A' still has all entries in m, and the invertibility of B has not been changed. Suppose that

$$A' = \begin{pmatrix} a & \rho \\ \gamma & A_0 \end{pmatrix}$$

where $a \in R$ (technically a is a 1×1 matrix over R), ρ is $1 \times (n-1)$, γ is $(n-1) \times 1$, and A_0 is $(n-1) \times (n-1)$. Then

$$xI_n = A'B' = \begin{pmatrix} a(1) + \rho(0) & a(0) + \rho B_0 \\ \gamma(1) + A_0(0) & \gamma(0) + A_0 B_0 \end{pmatrix} = \begin{pmatrix} a & \rho B_0 \\ \gamma & A_0 B_0 \end{pmatrix}$$

from which we can conclude that $xI_{n-1} = A_0B_0$. By the induction hypothesis, B_0 is invertible, and so B' is invertible, and the invertibility of B follows as well. $\square$

The following is critical in proving that if K has finite projective dimension over (R, m, K) then R is regular.

Theorem. *If M is finitely generated and has finite projective dimension over the local ring $(R,\, m,\, K)$, and $x \in m - m^2$ kills M and is not a zerodivisor in R, then M has finite projective dimension over R/xR.*

Proof. We may assume M is not 0. M cannot be free over R, since $xM = 0$. Thus, we may assume $\mathrm{pd}_R M \geq 1$. We want to reduce to the case where $\mathrm{pd}_R M = 1$. If $\mathrm{pd}_R M > 1$, we can think of M as a module over R/xR and map $(R/xR)^{\oplus h} \twoheadrightarrow M$ for some h. The kernel M_1 is a first module of syzygies of M over R/xR. By part (b) of the second Theorem on p. 5, $\mathrm{pd}_R M_1 = \mathrm{pd}_R M - 1$. Clearly, if M_1 has finite projective dimension over R/xR, so does M. By induction on $\mathrm{pd}_R M$ we have therefore reduced to the case where $\mathrm{pd}_R M = 1$. To finish the proof, we shall show that if $x \in m - m^2$ is not a zerodivisor in R, $xM = 0$, and $\mathrm{pd}_R M = 1$, then M is free over R/xR.

Consider a minimal free resolution of M over R, which will have the form

$$0 \to R^n \xrightarrow{A} R^k \to M \to 0$$

where A is an $k \times n$ matrix with entries in m. If we localize at x, we have $M_x = 0$, and so

$$0 \to R_x^n \to R_x^k \to 0$$

is exact. Thus, $k = n$, and A is $n \times n$. Let e_j denote the j th column of the identity matrix I_n. Since $xM = 0$, every xe_j is in the image of A, and so we can write $xe_j = Ab_j$ for a certain $n \times 1$ column matrix b_j over R. Let B denote the $n \times n$ matrix over R whose columns are $b_1, \ldots, b_n$. Then $xI_n = AB$. By the preceding Lemma, B is invertible, and so A and $AB = xI_n$ have the same cokernel, up to isomorphism. But the cokernel of xI_n is $(R/xR)^{\oplus n} \cong M = \mathrm{Coker}\,(A)$, as required. $\square$

We can now prove the result that we are aiming for, which completes the proof of the Auslander-Buchsbaum-Serre Theorem (p. 3).

Theorem. *Let (R, m, K) be a local ring such that $\mathrm{pd}_R K$ is finite. Then R is regular.*

Proof. If $m \in \mathrm{Ass}\,(R)$, then we find that K is free. But $K \cong R^n$ implies that $n = 1$ and R is a field, as required. We use induction on $\dim\,(R)$. The case where $\dim\,(R) = 0$ follows, since in that case $m \in \mathrm{Ass}\,(R)$.

Now suppose that $\dim\,(R) \geq 1$ and $m \notin \mathrm{Ass}\,(R)$. Then m is not contained in m^2 nor any of the primes in $\mathrm{Ass}\,(R)$, and so we can choose $x \in m$ not in m^2 nor in any associated prime. This means that x is not a zerodivisor in R. By the preceding Theorem, the fact that K has finite projective dimension over R implies that it has finite projective dimension over R/xR. By the induction hypothesis, R/xR is regular. Since $x \notin m^2$ and x is not a zerodivisor, both the least number of generators of the maximal ideal and the Krull dimension drop by one when we pass from R to R/xR. Since R/xR is regular, so is R. $\square$

We have finally proved the result we were aiming for, and we have now completed the argument given much earlier that a localization of a regular local ring is regular. We also note:

Theorem. *Let $R \to S$ be a faithfully flat homomorphism of Noetherian rings. If S is regular, then R is regular.*

Proof. Let P be a maximal ideal of R. Then $PS \neq S$, and there is a maximal ideal Q of S lying over P. It suffices to show that every R_P is regular, and we have that $R_P \to S_Q$ is flat and local. Thus, we have reduced to the case where R and S is local and the map is local. Take a minimal free resolution of R/P over R. If R is not regular, this resolution is infinite. Apply $S \otimes_R \_$. Since S is R-flat, we get a free resolution of S/PS over S. Since P maps into Q, this resolution is still minimal. Thus, S/PS has infinite projective dimension over S, contradicting the fact that S is regular. $\square$

Grothendieck groups and unique factorization in regular rings

we introduce Grothendieck groups and use them to prove that regular local rings are unique factorization domains, following M. P. Murthy.

Let R be a Noetherian ring. Let $\mathcal{M}$ denote the set of modules

$$\{R^n/M : n \in \mathbb{N},\, M \subseteq R^n\}.$$

Every finitely generated R-module is isomorphic to one in $\mathcal{M}$, which is all that we really need about $\mathcal{S}$: we can also start with some other set of modules with this property without affecting the Grothendieck group, but we use this one for definiteness.

Consider the free abelian group with basis $\mathcal{M}$, and kill the subgroup generated by all elements of the form $M - M' - M''$ where

$$0 \to M' \to M \to M'' \to 0$$

is a short exact sequence of elements of $\mathcal{M}$. The quotient group is called the *Grothendieck group* $G_0(R)$ of R. It is an abelian group generated by the elements $[M]$, where $[M]$ denotes the image of $M \in \mathcal{M}$ in $G_0(R)$. Note that if $M' \cong M$ we have a short exact sequence

$$0 \to M' \to M \to 0 \to 0,$$

so that $[M] = [M'] + [0] = [M']$, i.e., isomorphic modules represent the same class in $G_0(R)$.

A map L from $\mathcal{M}$ to an abelian group $(A, +)$ is called *additive* if whenever

$$0 \to M' \to M \to M'' \to 0$$

is exact, then $L(M) = L(M') + L(M'')$. The map γ sending M to $[M] \in G_0(R)$ is additive, and is a universal additive map in the following sense: given any additive map $L : \mathcal{M} \to A$, there is a unique homomorphism $h : G_0(M) \to A$ such that $L = h \circ \gamma$. Since we need $L(M) = h([M])$, if there is such a map it must be induced by the map from the free abelian group with basis $\mathcal{M}$ to A that sends M to $h(M)$. Since h is additive, the elements $M - M' - M''$ coming from short exact sequences

$$0 \to M' \to M \to M'' \to 0$$

are killed, and so there is an induced map $h : G_0(R) \to A$. This is obviously the only possible choice for h.

Over a field K, every finitely generated module is isomorphic with $K^{\oplus n}$ for some $n \in \mathbb{N}$. It follows that $G_0(K)$ is generated by $[K]$, and in fact it is $\mathbb{Z}[K]$, the free abelian group on one generator. The additive map associated with the Grothendieck group sends M to $\dim_K(M)[K]$. If we identify $\mathbb{Z}[K]$ with $\mathbb{Z}$ by sending $[K]$ to 1, this is the dimension map.

If R is a domain with fraction field $\mathcal{F}$, we have an additive map to $\mathbb{Z}$ that sends M to $\dim_{\mathcal{F}}\mathcal{F} \otimes_R M$, which is called the *torsion-free rank* of M. This induces a surjective map $G_0(R) \to \mathbb{Z}$. If R is a domain and $[R]$ generates $G_0(R)$, then $G_0(R) \cong \mathbb{Z}[R] \cong \mathbb{Z}$, with the isomorphism given by the torsion-free rank map.

Notice that if L is additive and

$$0 \to M_n \to \cdots \to M_1 \to M_0 \to 0$$

is exact, then

$$L(M_0) - L(M_1) + \cdots + (-1)^n L(M_n) = 0.$$

If $n \leq 2$, this follows from the definition. We use induction. In the general case note that we have a short exact sequence

$$0 \to N \to M_1 \to M_0 \to 0$$

and an exact sequence

$$0 \to M_n \to \cdots \to M_3 \to M_2 \to N \to 0,$$

since

$$\mathrm{Coker}\,(M_3 \to M_2) \cong \mathrm{Ker}\,(M_1 \to M_0) = N.$$

Then

$$(*) \quad L(M_0) - L(M_1) + L(N) = 0,$$

and

$$(**) \quad L(N) - L(M_2) + \cdots + (-1)^{n-1} L(M_n) = 0$$

by the induction hypothesis. Subtracting $(**)$ from $(*)$ yields the result. $\square$

From these comments and our earlier results on regular local rings we get at once:

Theorem. *If R is a regular local ring, $G_0(R) = \mathbb{Z}[R] \cong \mathbb{Z}$.*

Proof. R is a domain, and we have the map given by torsion-free rank. It will suffice to show that $[R]$ generates $G_0(R)$. But if M is any finitely generated R-module, we know that M has a finite free resolution

$$0 \to R^{b_k} \to \cdots \to R^{b_1} \to R^{b_0} \to M \to 0,$$

and so the element $[M]$ may be expressed as

$$[R^{b_0}] - [R^{b_1}] + \cdots + (-1)^k [R^{b_k}] = b_0[R] - b_1[R] + \cdots + (-1)^k b_k[R] = (b_0 - b_1 + \cdots + (-1)^k b_k)[R]$$

$\square$

Note that given a finite filtration

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_{n-1} \subseteq M_n = M$$

of a finitely generated R-module M and an additive map L we have that

$$L(M) = L(M_n/M_{n-1}) + L(M_{n-1}),$$

and, by induction on n, that

$$L(M) = \sum_{j=1}^{n} L(M_j/M_{j-1}).$$

In particular, $[M] \in G_0(R)$ is

$$\sum_{j=1}^{n} [M_j/M_{j-1}].$$

Theorem. *Let R be a Noetherian ring. $G_0(R)$ is generated by the elements $[R/P]$, as P runs through all prime ideals of R. If P is prime and $x \in R - P$, then $[R/(P + xR)] = 0$, and so if $R/Q_1, \ldots, R/Q_k$ are all the factors in a prime filtration of $[R/(P + xR)]$, we have that $[R/Q_1] + \cdots + [R/Q_k] = 0$. The relations of this type are sufficient to generate all relations on the classes of the prime cyclic modules.*

Proof. The first statement follows from the fact that every finitely generated module over a Noetherian ring R has a finite filtration in which the factors are prime cyclic modules. The fact that $[R/(P + xR)] = 0$ follows from the short exact sequence

$$0 \to R/P \xrightarrow{x} R/P \to R/(P + xR) \to 0,$$

which implies $[R/P] = [R/P] + [R/(P + xR)]$ and so $[R/(P + xR)] = 0$ follows.

Now, for every $M \in \mathcal{M}$, fix a prime cyclic filtration of M. We need to see that if we have a short exact sequence

$$0 \to M' \to M \to M'' \to 0$$

that the relation $[M] = [M'] + [M'']$ is deducible from ones of the specified type. We know that M' will be equal to the sum of the classes of the prime cyclic module coming from its chosen prime filtration, and so will M''. These two prime cyclic filtrations together induce a prime cyclic filtration $\mathcal{F}$ of M, so that the information $[M] = [M'] + [M'']$ is conveyed by setting $[M]$ equal to the sum of the classes of the prime cyclic modules in these specified filtrations of $[M]$ and $[M']$. But $\mathcal{F}$ will not typically by the specified filtration of $[M]$, and so we need to set the sum of the prime cyclic modules in the specified filtration of M equal to the sum of all those occurring in the specified filtrations of M' and M''.

Thus, we get all relations needed to span if for all finitely generated modules M and for all pairs of possibly distinct prime cyclic filtrations of M, we set the sum of the classes of the prime cyclic modules coming from one filtration equal to the corresponding sum for the other. But any two filtrations have a common refinement. Take a common refinement, and refine it further until it is a prime cyclic filtration again. Thus, we get all relations needed to span if for every finitely generated module M and for every pair consisting of a prime cyclic filtration of M and a refinement of it, we set the sum of the classes coming from one filtration to the sum of those in the other. Any two prime cyclic filtrations may then be compared by comaring each two a prime cyclic filtration that refines them both.

In refining a given prime cyclic filtration, each factor R/P is refined. Therefore, we get all relations needed to span if for every R/P and every prime cyclic filtration of R/P, we set $[R/P]$ equal to the sum of the classes in the prime cyclic filtration of R/P. Since $\mathrm{Ass}\,(R/P) = P$, the first submodule of a prime cyclic filtration of R/P will be isomorphic with R/P, and will therefore have the form $x(R/P)$, where $x \in R - P$. If the other factors are $R/Q_1, \ldots, R/Q_k$, then these are the factors of a filtration of $(R/P)/x(R/P) = R/(P + xR)$. Since $[x(R/P)] = [R/P]$, the relation we get is

$$[R/P] = [R/P] + [R/Q_1] + \cdots + [R/Q_k],$$

which is equivalent to

$$[R/Q_1] + \cdots + [R/Q_k] = 0,$$

and so the specified relations suffice to span all relations. $\square$

Corollary. $G_0(R) \cong G_0(R_{\mathrm{red}})$.

Proof. The primes of R_{red} and those of R are in bijective correspondence, and the generators and relations on them given by the preceding Proposition are the same. $\square$

Proposition. *If R and S are Noetherian rings, then $G_0(R \times S) \cong G_0(R) \times G_0(S)$.*

Proof. If M is an $(R \times S)$-module, then with $e = (1, 0)$ and $f = (0, 1)$ we have an isomorphism $M \cong eM \times fM$, where eM is an R-module via $r(em) = (re)(em)$ and fM is an S-module via $s(fm) = (sf)(fm)$. There is an isomorphism $M \cong eM \times fM$. Conversely, given an R-module A and an S-module B, these determine an $R \times S$-module $M = A \times B$, where $(r, s)(a, b) = (ra, sb)$ such that $eM \cong A$ over R and $fM \cong B$ over R. Thus, $(R \times S)$-modules correspond to pairs A, B where A is an R-module and B is an S-module. Moreover, if $h : M \to M'$ then h induces maps $eM \to eM'$ and $fM \to fM'$ that determine h. Said differently, a map from $A \times B \to A' \times B'$ as $(R \times S)$-modules corresponds to a pair of maps $A \to A'$ as R-modules and $B \to B'$ as S-modules. Consequently, a short exact sequence of $(R \times S)$-modules corresponds to a pair consisting of short exact sequences, one of R-modules and the other of S-modules. The stated isomorphism of Grothendieck groups follows at once. $\square$

Proposition. *Let R be an Artin ring.*

(a) *If (R, m, K) is Artin local, $G_0(R) \cong \mathbb{Z} \cdot [K] \cong \mathbb{Z}$, where the additive map $M \mapsto \ell_R(M)$ gives the isomorphism with $\mathbb{Z}$.*

(b) *If R has maximal ideals $m_1, \ldots, m_k$, then $G_0(R)$ is the free abelian group on the $[R/m_j]$.*

Proof. For part (b), notice that the R/m_k are generators by Theorem, and there are no non-trivial relations, since if $x \notin m_j$, $R/(m_j + xR) = 0$. Part (a) follows easily from part (b). We may also deduce part (b) from part (a), using the fact that an Artin ring is a finite product of Artin local rings and the preceding Proposition. $\square$

Proposition. *Let R and S be Noetherian rings.*

(a) *If $R \to S$ is a flat homomorphism, there is a a group homomorphism $G_0(R) \to G_0(S)$ sending $[M]_R \mapsto [S \otimes_R M]_S$. Thus, G_0 is a covariant functor from the category of rings and flat homomorphisms to abelian groups.*

(b) *If $S = W^{-1}R$ is a localization, the map described in (a) is surjective.*

(c) *If P is a minimal prime of R, there is a homomorphism $G_0(R) \to \mathbb{Z}$ given by $[M] \mapsto \ell_{R_P}(M_P)$. Of course, if R is a domain and $P = (0)$, this is the torsion-free rank map.*

(d) *If R is a domain, the map $\mathbb{Z} \to G_0(R)$ that sends 1 to $[R]$ is split by the torsion-free rank map. Thus, $G_0(R) = \mathbb{Z}[R] + \overline{G}_0(R)$, where $\overline{G}_0(R) = G_0(R)/\mathbb{Z} \cdot [R]$, the reduced Grothendieck group of R. When R is a domain, the reduced Grothendieck group may be thought of as the subgroup of $G_0(R)$ spanned by the classes of the torsion R-modules.*

(e) *If S is module-finite over R, there is a group homomorphism $G_0(S) \to G_0(R)$ sending $[M]_S$ to $[{}_RM]_R$, where ${}_RM$ denotes M viewed as an R-module via restriction of scalars. In particular, this holds when S is homomorphic image of I. Thus, G_0 is a contravariant functor from the category of rings and module-finite homomorphisms to abelian groups.*

Proof. (a) is immediate from the fact that $S \otimes_R \_$ preserves exactness.

To prove (b), note that if M is a finitely generated module over $W^{-1}R$, it can be written as the cokernel of a matrix of the form (r_{ij}/w_{ij}), where every $r_{ij} \in R$ and every $w_{ij} \in W$. Let w be the product of all the w_{ij}. Then the entries of the matrix all have the form r'_{ij}/w. If we multiply every entry of the matrix by w, which is a unit in S, the cokernel is unaffected: each column of the matrix is multiplied by a unit. Let $M_0 = \text{Coker }(r'_{ij})$. Then $S \otimes_R M_0 \cong M$. This shows the surjectivity of the map of Grothendieck groups.

Part (c) is immediate from the fact that localization is exact coupled with the fact the length is additive. The statement in (d) is obvious, since the torsion-free rank of R is 1.

One has the map in (e) because restriction of scalars is an exact functor from finitely generated S-modules to finitely generated R-modules. One needs that S is module-finite over R to guarantee that when one restricts scalars, a finitely generated S-module becomes a finitely generated R-module. $\square$

If S is faithfully flat or even free over R, the induced map $[M]_R \to [S \otimes_R M]_S$ need not be injective, not even if $S = L \otimes_K R$ where L is a finite field extension of $K \subseteq R$: an example is given in the sequel: see the top of p. 16. An R-module M is said to have *finite Tor dimension* or *finite flat dimension* over R at most d if $\text{Tor}_i^R(M, N) = 0$ for all $i > d$. If $M = 0$, the Tor dimension is defined to be -1. Otherwise, it is the smallest integer d such that $\text{Tor}_i^R(M, N) = 0$ for all $i > d$, if such an integer exists, and $+\infty$ otherwise. We leave it as an exercise to show that M has finite Tor dimension at most d if and only if some (equivalently, every) d th module of syzygies of M is flat. Likewise, M has finite Tor dimension at most d if and only if M has a left resolution by flat modules of length at most d. A nonzero module M has Tor dimension 0 if and only of M is flat over R. Of course, if M has finite projective dimension d, then M has Tor dimension at most d.

Proposition. *If S is a Noetherian R-algebra of finite Tor dimension $\leq d$ over the Noetherian ring R, there is a map $G_0(R) \to G_0(S)$ that sends $[M]_R$ to*

$$\theta(M) = \sum_{i=0}^{d} (-1)^i [Tor_i^R(S, M)]_S.$$

Proof. We simply need to check the additivity of the map. Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of finitely generated R-modules. Then we get a long exact sequence of finitely generated S-modules

$$0 \to \text{Tor}_d^R(S, M') \to \text{Tor}_d^R(S, M) \to \text{Tor}_d^R(S, M'') \to \cdots$$

$$\to \mathrm{Tor}_0^R(S,\, M') \to \mathrm{Tor}_0^R(S,\, M) \to \mathrm{Tor}_0^R(S,\, M'') \to 0$$

and so the alternating sum Σ of the classes of these modules in $G_0(S)$ is 0. We think of these $3d$ modules as in positions $3d-1,\, 3d-2,\, \cdots, 2,\, 1,\, 0$ counting from the left. The terms involving M'' are in positions numbered $0,\ ,3,\ 6,\ \ldots\ ,\ 3(d-1)$. Their signs alternate starting with $+$, and so their contribution to Σ is $\theta(M'')$. The terms involving M are in positions numbered $1,\ 4,\ 7,\ \ldots\ ,\ 3(d-1)+1$. Their signs alternate starting with $-$, and so their contribution to Σ is $-\theta(M)$. Finally, the terms involving M' are in positions numbered $2,\ 5,\ 8,\ \ldots\ ,\ 3(d-1)+2$. Their signs alternate starting with $+$, and so their contribution to Σ is $\theta(M')$. This yields $0 = \Sigma = \theta(M') - \theta(M) + \theta(M'')$, as required. □

Corollary. *If x is not a zerodivisor in the Noetherian ring R, there is a map $G_0(R) \to G_0(R/xR)$ that sends $[M]_R \mapsto [M/xR]_{R/xR} - [\mathrm{Ann}_M\, x]_{R/xR}$.*

Proof. This is the special case of result just above when $S = R/xR$, which has projective dimension at most 1 and, hence, flat dimension at most 1. We have that $\mathrm{Tor}_0^R(R/xR,\, M) \cong (R/xR) \otimes_R M \cong M/xM$, and $\mathrm{Tor}_1^R(R/xR,\, M) \cong \mathrm{Ann}_M x$. An elementary proof of this result may be given by showing that when

$$0 \to M' \to M \to M'' \to 0$$

is exact then so is

$$0 \to \mathrm{Ann}_{M'}\, x \to \mathrm{Ann}_M\, x \to \mathrm{Ann}_{M''}\, x \to M'/xM' \to M/xM \to M''/xM'' \to 0,$$

developing this special case of the long exact sequence for Tor from first principles. □

Corollary. *Let R be Noetherian and let S denote either $R[x]$ or $R[[x]]$, where x is an indeterminate. Since S is flat over R, we have an induced map $G_0(R) \to G_0(S)$. This map is injective.*

Proof. We have that $S/xS \cong R$, where x is not a zerodivisor in S, and so we have a map $G_0(S) \to G_0(R)$. Under the composite map, the class $[M]_R$ of an R-module M maps first to $[M[x]\,]_S$ (respectively, $[M[[x]]\,]_S$), and then to $[M[x]/xM[x]\,]_R$ (respectively, $[M[[x]]/xM[[x]]\,]_R$), since x is not a zerodivisor on $M[x]$ (respectively, $M[[x]]$). In both cases, the quotient is $\cong M$, and so the composite map takes $[M]_R \to [M]_R$. Thus, the composite $G_0(R) \to G_0(S) \to G_0(R)$ is the identity on $G_0(R)$, which implies that $G_0(R) \to G_0(S)$ is injective. □

We next aim to establish the following result, which will imply unique factorization in regular local rings.

Theorem (M. P. Murthy). *Let R be a normal domain and let H be the subgroup of $\overline{G}_0(R)$ spanned by the classes $[R/P]$ for P a prime of height 2 or more. Then*

$$\mathcal{C}\ell\,(R) \cong \overline{G}_0(R)/H.$$

Assuming this for the moment, note the failure of the injectivity of the map from $G_0(R) \to G_0(S)$ where $R = \mathbb{R}[x, y]/(x^2 + y^2 - 1)$ and $S = \mathbb{C} \otimes_{\mathbb{R}} R \cong \mathbb{C}[x, y]/(x^2 + y^2 - 1)$. We have seen in Problem **4.** of Problem Set #5 from Math 614, Fall 2013, that the maximal ideal $P = (x - 1, y)R$ is not principal in R, from which it will follow that $[P]$ is nonzero in $\mathcal{C}\ell\,(R)$, and so that $[R/P]$ is nonzero in $\overline{G}_0(R)$, and therefore $[R/P]$ is not zero in $G_0(R)$. But P becomes principal when expanded to S. In fact, S is a UFD, for if we let $u = x + yi$ and $v = x - yi$, then $\mathbb{C}[x, y] \cong \mathbb{C}[u, v]$ (we have made a linear change of variables), and so $S \cong \mathbb{C}[u, v]/(uv - 1) \cong \mathbb{C}[u][1/u]$. Thus, $[S \otimes R/P]_S = [S/PS]_S = 0$ in $G_0(S)$.

Recall that if R is a normal domain, one defines the *divisor class group* of R, denoted $\mathcal{C}\ell\,(R)$, as follows. First form the the free abelian group on generators in bijective correspondence with the height one prime ideals of R. The elements of this group are called *divisors*. The divisor $\mathrm{div}\,(I)$ of an ideal $I \neq 0$ whose primary decomposition only involves height one primes (I is said to be of *pure height one*) is then obtained from the primary decomposition of I: if the primary decomposition of I is $P_1^{(k_1)} \cap \cdots \cap P_s^{(k_s)}$ where the P_j are mutually distinct, then $\mathrm{div}\,(I) = \sum_{j=1}^{s} k_j P_j$. We regard the unit ideal as having pure height one in a vacuous sense, and define its divisor to be 0. The divisor $\mathrm{div}\,(r)$ of an element $r \in R - \{0\}$ is the divisor of rR, and, hence, 0 if r is a unit. Then $\mathcal{C}\ell\,(R)$ is the quotient of the free abelian group of divisors by the span of the divisors of nonzero principal ideals. The following is part of a Theorem p. 137 of the Math 614 Lecture Notes for Fall 2013, to which we refer the reader for the proof.

Theorem. *Let R be a Noetherian normal domain. If I is pure height one, then so is fI for every nonzero element f of R, and* $\mathrm{div}\,(fI) = \mathrm{div}\,(f) + \mathrm{div}\,(I)$. *For any two ideals I and J of pure height one,* $\mathrm{div}\,(I) = \mathrm{div}\,(J)$ *iff $I = J$, while the images of* $\mathrm{div}\,(I)$ *and* $\mathrm{div}\,(J)$ *in $\mathcal{C}\ell\,(R)$ are the same iff there are nonzero elements f, g of R such that $fI = gJ$. This holds iff I and J are isomorphic as R-modules. In particular, I is principal if and only if* $\mathrm{div}\,(I)$ *is 0 in the divisor class group. Hence, R is a UFD if and only if $\mathcal{C}\ell\,(R) = 0$.*

While we are not giving a full proof here, we comment on one point. If $I \cong J$ as an R-module, the isomorphism is given by an element of $\mathrm{Hom}_R(I, J)$. If we localize at the prime (0), which is the same as applying $\mathcal{F} \otimes_R \_$, where $\mathcal{F}$ is the fraction field of R, we see that $\mathrm{Hom}_R(I, J)$ embeds in $\mathcal{F} \otimes_R \mathrm{Hom}_R(I, J) \cong \mathrm{Hom}_{\mathcal{F}}(I\mathcal{F}, J\mathcal{F}) = \mathrm{Hom}\mathcal{F}(\mathcal{F}, \mathcal{F}) \cong \mathcal{F}$, that is, every homomorphism from I to J is induced by multiplying by a suitable fraction f/g, $f \in R$, $g \in R - \{0\}$. When this fraction gives an isomorphism we have $(f/g)I = J$ or $fI = gJ$.

Theorem (M. P. Murthy). *Let R be a normal domain and let H be the subgroup of $\overline{G}_0(R)$ spanned by the classes $[R/P]$ for P prime of height 2 or more. Then $\mathcal{C}\ell\,(R) \cong \overline{G}_0(R)/H$ with the map sending $[P] \mapsto [R/P]$ for all height one primes P.*

Before proving this, we note two corollaries. One is that regular local rings have unique factorization. Whether this is true was an open question for many years that was first settled by M. Auslander and D. Buchsbaum by a much more difficult method, utilizing homological methods but based as well on a result of Zariski that showed it suffices to prove the result for regular local rings of dimension 3. Later, I. Kaplansky gave a substantially simpler proof. But I feel that Murthy's argument gives the "right" proof. We have already seen that for a regular local ring R, $\overline{G}_0(R) = 0$. Therefore:

Corollary. *A regular local ring is a UFD.* $\square$

Corollary. *If R is a Dedekind domain, then $\overline{G}_0(R) \cong \mathcal{C}\ell\,(R)$ and $G_0(R) = \mathbb{Z}\cdot[R] \oplus \mathcal{C}\ell\,(R)$.*

Proof. This is clear, since there are no primes of height two or more. $\square$

We now go back and prove Murthy's result.

Proof of the Theorem. We know that $G_0(R)$ is the free group on the classes of the R/P, P prime, modulo relations obtained from prime cyclic filtrations of $R/(P+xR)$, $x \notin P$. We shall show that if we kill $[R]$ and all the $[R/Q]$ for Q of height 2 or more, all relations are also killed except those coming from $P = (0)$, and the image of any relation corresponding to a prime cyclic filtration of R/xR corresponds precisely to $\mathrm{div}\,(x)$. Clearly, if $P \neq 0$ and $x \notin P$, any prime containing $P + xR$ strictly contains P and so has height two or more. Thus, we need only consider relations on the R/P for P of height one coming from prime cyclic filtrations of R/xR, $x \neq 0$. Clearly, R does not occur, since R/xR is a torsion module, and occurrences of R/Q for Q of height ≥ 2 do not matter. We need only show that for every prime P of height one, the number of occurrences of R/P in any prime cyclic filtration of R/xR is exactly k, where $P^{(k)}$ is the P-primary component of xR. But we can do this calculation after localizing at P: note that all factors corresponding to other primes become 0, since some element in the other prime not in P is inverted. Then $xR_P = P^kR_p$, and we need to show that any prime cyclic filtration of R_P/xR_P has k copies of R_P/PR_P, where we know that $xR_P = P^kR_P$. Notice that $(R_P,\, PR_P)$ is a DVR, say (V, tV), and $xR_P = t^kV$. The number of nonzero factors in any prime cyclic filtration of V/t^kV is the length of V/t^kV over V, which is k, as required: the only prime cyclic filtration without repetitions is

$$0 \subset t^{k-1}V \subset t^{k-2}V \subset \cdots \subset t^2V \subset tV \subset V. \quad \square$$

Theorem. *$G_0(R) \cong G_0(R[x])$ under the map that sends $[M] \mapsto [M[x]]$, where we have written $M[x]$ for $R[x] \otimes_R M$.*

Proof. We have already seen that the map is injective, and even constructed a left inverse for it, which takes

$$[N]_{R[x]} \mapsto [N/xN]_R - [\mathrm{Ann}_N\, x]_R.$$

However, we shall not make use of this left inverse to prove surjectivity. Instead, we prove that every $[S/Q]$, Q prime, is in the image of $G_0(R) \to G_0(R[x])$ by Noetherian induction on $R/(Q \cap R)$. There are two sorts of primes lying over $P \in \mathrm{Spec}\,(R)$. One is $PR[x]$. The other is generated, after localization at $R - P$, by a polynomial $f \in R[x]$ of positive degree with leading coefficient in $R - P$ such that the image of f is irreducible in $\kappa_P[x]$, where $\kappa_P = R_P/PR_P \cong \mathrm{frac}\,(R/P)$. To see this, note that every prime Q lying over P corresponds, via contraction to $R[x]$, to a prime of the fiber $(R-P)^{-1}(R/P)[x] \cong \kappa_P[x]$. The primes in $\kappa_P[x]$ are of two types: there is the (0) ideal, whose contraction to $R[x]$

is $PR[x]$, and there are the maximal ideals, each of which is generated by an irreducible polynomial of positive degree in $\kappa_P[x]$. We can clear the denominators by multiplying by an element of $R-P$, and then lift the nonzero coefficients to $R-P$, to obtain a polynomial f with leading coefficient in $R-P$ as described previously. Note that Q is recovered from P and f as the set of all elements of $R[x]$ multiplied into $P+fR[x]$ by an element of $R-P$. Briefly, $Q = (PR[x] + fR[x]) :_{R[x]} (R-P)$.

Since $R[x]/PR[x] = (R/P) \otimes_R R[x]$ is evidently in the image, we need only show that the primes Q of the form $(PR[x]+fR[x]) :_{R[x]} (R-P)$ are in the image of $G_0(R) \to G_0(R[x])$. We have exact sequences

$$(*) \quad 0 \to (R/P)[x] \xrightarrow{f} (R/P)[x] \to M \to 0,$$

where $M = R[x]/(PR[x] + fR[x])$, and

$$(**) \quad N \to M \to R[x]/Q \to 0.$$

Because $(R-P)^{-1}M = (R-P)^{-1}R[x]/Q$, we have that N is a finitely generated module that is a torsion module over R/P. Since every generator of N is killed by an element of $R-P$, we can choose $a \in R-P$ that kills N. From $(*)$, $[M] = 0$ in $G_0(S)$. From $(**)$, $[R[x]/Q] = -[N]$ in $G_0(R[x])$. Therefore, it suffices to show that $[N]$ is in the image. In a prime cyclic filtration of N, every factor is killed by $P + aR$, and therefore for every $R[x]/Q'$ that occurs, Q' lies over a prime strictly containing P. But then every $[R[x]/Q']$ is in the image by the hypothesis of Noetherian induction. $\square$

Theorem. *Let R be a ring and S a multiplicative system. Then the kernel of $G_0(R) \to G_0(S^{-1}R)$ is spanned by the set of classes $\{[R/P] : P \cap S \neq \emptyset\}$. Hence, for any $x \in R$ there is an exact sequence*

$$G_0(R/xR) \to G_0(R) \to G_0(R_x) \to 0.$$

Proof. The final statement is immediate from the general statement about localization at S, since $G_0(R/xR)$ is spanned by classes $[R/P]_{R/xR}$ such that $x \in P$ and $x \in P$ iff P meets $\{x^n : n \geq 1\}$, and so the image of $G_0(R/xR)$ in $G_0(R)$ is spanned by the classes $[R/P]_R$ for $x \in P$.

To prove the general statement about localization, first note that the specified classes are clearly in the kernel. To show that these span the entire kernel, it suffices to show that all the spanning relations on the classes $[S^{-1}R/QS^{-1}R_Q]$ hold in the quotient of $G_0(R)$ by the span Γ of the classes $[R/P]$ for $P \cap S \neq \emptyset$. Consider a prime cyclic filtration of $S^{-1}R/(PS^{-1}R + (x))$, where x may be chosen in R. We may contract (i.e., take inverse images of) the submodules in this filtration to get a filtration of R/P. Each factor N_i contains an element u_i such that, after localization at S, u_i generates $S^{-1}N_i \cong S^{-1}R/Q_i$. Thus, for each i, we have short exact sequences

$$0 \to Ru_i \to N_i \to C_i \to 0 \qquad \text{and} \qquad 0 \to D_i \to Ru_i \to R/Q_i \to 0,$$

where C_i and D_i vanish after localization at S and so have prime cyclic filtrations with factors $R/\mathcal{Q}_j$ such that $\mathcal{Q}_j$ meets S. Here, we have that $Q_1 = P$. We must show that the relation $\sum_{i>1}[S^{-1}R/S^{-1}Q_i] = 0$ comes from a relation on the $[R/Q_i]$ in $G_0(R)/\Gamma$. But $[R/P] = N_1 + \sum_{i>1}[N_i]$, and for every i,

$$[N_i] = [Ru_i] + [C_i] = [R/Q_i] + [C_i] + [D_i].$$

Since $Q_1 = P$, we have

$$0 = [C_1] + [D_1] + \sum_{i>1}[R/Q_i] + [C_i] + [D_i]$$

in $G_0(R)$, and the conclusion we want follows: as aleady observed, every C_i and every D_i is killed by an element of S, and so has a prime cyclic filtration in which each prime cyclic module has a class in Γ. $\square$

We next define the *Grothendieck group of projective modules* over a Noetherian ring R by forming the free abelian group on generators P in $\mathcal{M}$ (one can work with any set of finitely generated projective modules containing a representative of every isomorphism class) and killing the subgroup spanned by elements $P - P' - P''$, where $0 \to P' \to P \to P'' \to 0$ is exact. In this situation the short exact sequence of projectives is split (this only uses that P'' is projective), and so $P \cong P' \oplus P''$. Thus, the elements that we kill to construct $K_0(R)$ have the form $(P' \oplus P'') - P' - P''$. Note that isomorphic projectives represent the same class in $K_0(R)$.

There is obviously a canonical map $K_0(R) \to G_0(R)$ that takes $[P]$ in $K_0(R)$ to $[P]$ in $G_0(R)$ for every finitely generated projective module over R.

Theorem. *If R is regular, the map $K_0(R) \cong G_0(R)$ is an isomorphism.*

Proof. We want to define a map from $G_0(R)$ to $K_0(R)$. Given a finitely generated R-module M, we can choose a finite projective resolution of M by finitely generated projective modules, say $P_\bullet$, and suppose that the length of this resolution is d. The obvious way to define an inverse map is to send $[M]$ to

$$[P_0] - [P_1] + \cdots + (-1)^d[P_d] \in K_0(R).$$

We must check that this is independent of the choice of the projective resolution. Given another such projective resolution $Q_\bullet$ of M we must show that the two alternating sums are the same in $K_0(R)$ (this is obvious in $G_0(R)$, since both equal $[M]$, but M is not "available" in $K_0(R)$). To prove this, choose a map of complexes $\phi_\bullet : P_\bullet \to Q_\bullet$ such that the induced map of augmentations $M = H_0(P_\bullet) \to H_0(Q_\bullet) = M$ is the identity. Form $C_\bullet$, the mapping cone of ϕ, which is a complex of projective modules. Then $C_n = P_n \oplus Q_{n-1}$. We claim that $C_\bullet$ is exact (not just acyclic): *all* the homology vanishes. To see this, consider the long exact sequence of the mapping cone:

$$\cdots \to H_n(Q_\bullet) \to H_n(C_\bullet) \to H_{n-1}(P_\bullet) \to H_{n-1}(Q_\bullet) \to \cdots .$$

If $n \geq 2$, $H_n(C_\bullet) = 0$ since $H_n(Q_\bullet)$ and $H_{n-1}(P_\bullet)$ both vanish. If $n = 1$, $H_1(C_\bullet)$ vanishes because $H_1(Q_\bullet) = 0$ and the connecting homomorphism $H_0(P_\bullet) \to H_0(Q_\bullet)$ is an isomorphism. If $n = 0$, $H_0(C_\bullet) = 0$ because $H_0(Q_\bullet)$ and $H_{-1}(P_\bullet)$ both vanish.

Thus, the alternating sum of the classes in $C_\bullet$ is 0 in $K_0(R)$, and this is exactly what we want.

Additivity follows because given a short exact sequence of finitely generated modules $0 \to M' \to M \to M'' \to 0$ and projective resolutions $P'_\bullet$ of M' and $P''_\bullet$ of M'' by finitely generated projective modules, one can construct such a resolution for M whose j th term is $P'_j \oplus P''_j$. See pp. 12 and 13 of the supplement entitled *The Functor Tor.* $\square$

Note that K_0 is a functor on all maps of Noetherian rings (not just flat maps) because short exact sequences of projectives are split and remain exact no matter what algebra one tensors with. Restriction of scalars from S to R will not induce a map on K_0 unless S is module-finite and *projective* over R.

Observe also that $K_0(R)$ has a commutative ring structure induced by $\_ \otimes_R \_$, with $[R]$ as the multiplicative identity, since the tensor product of two finitely generated projective modules is a projective module, and tensor distributes over direct sum.

Proposition. *Let P and Q be finitely generated projective modules over a Noetherian ring R. Then $[P] = [Q]$ in $K_0(R)$ if and only there is a free module G such that $P \oplus G \cong Q \oplus G$.*

Proof. $[P] = [Q]$ if and only if $[P] - [Q]$ is in the span of the standard relations used to define $K_0(R)$, in which case, for suitable integers h, k,

$$P - Q = \sum_{i=1}^{h} \big((P_i \oplus Q_i) - P_i - Q_i\big) + \sum_{j=1}^{k} \big(P'_j + Q'_j - (P'_j \oplus Q'_j)\big)$$

and so

$$P + \sum_{i=1}^{h}(P_i + Q_i) + \sum_{j=1}^{k}(P'_j \oplus Q'_j) = Q + \sum_{i=1}^{h}(P_i \oplus Q_i) + \sum_{j=1}^{k}(P'_j + Q'_j).$$

The fact that this equation holds implies that the number of occurrences of any given projective module on the left hand side is equal to the number of occurrences of that projective module on the right hand side. Therefore, if we change every plus sign (+) to a direct sum sign ($\oplus$), the two sides of the equation are isomorphic modules: the terms occurring in the direct sum on either side are the same except for order. Therefore:

$$P \oplus \bigoplus_{i=1}^{h}(P_i + Q_i) \oplus \bigoplus_{j=1}^{k}(P'_j \oplus Q'_j) = Q \oplus \bigoplus_{i=1}^{h}(P_i \oplus Q_i) \oplus \bigoplus_{j=1}^{k}(P'_j \oplus Q'_j).$$

In other words, if we let

$$N = \bigoplus_{i=1}^{h}(P_i \oplus Q_i) \oplus \bigoplus_{j=1}^{k}(P'_j \oplus Q'_j),$$

then $P \oplus N = Q \oplus N$. But N is projective, and so we can choose N' such that $N \oplus N' \cong G$ is a finitely generated free module. But then

$$P \oplus N \oplus N' \cong Q \oplus N \oplus N',$$

i.e., $P \oplus G \cong Q \oplus G$. $\square$

Corollary. *let R be Noetherian. $K_0(R)$ is generated by $[R]$ if and only if every projective module P has a finitely generated free complement, i.e., if and only if for every finitely generated projective module M there exist integers h and k in $\mathbb{N}$ such that $P \oplus R^h \cong R^k$.* $\square$

We know that

$$K_0(K[x_1, \dots, x_n]) \cong G_0(K[x_1, \dots, x_n]) \cong G_0(K) \cong \mathbb{Z}$$

is generated by the class of R. Therefore, every finitely generated projective module over $R = K[x_1, \dots, x_n]$ has a finitely generated free complement. To prove that every projective module over a R is free, it suffices to show that if $P \oplus R \cong R^n$ then $P \cong R^{n-1}$. The hypothesis implies precisely that P is the kernel of a map $R^n \twoheadrightarrow R$. Such a map is given by a $1 \times n$ matrix $(r_1 \ \dots \ r_n)$. The surjectivity of the map corresponds to the condition that the r_j generate the unit ideal of R. If $\sum_{j=1}^{n} r_j s_j = 1$, then the $n \times 1$ column matrix whose entries are $s_1, \dots, s_n$ mapping $R \to R^n$ gives a splitting. $P \cong R^{n-1}$ implies that this column vector v can be extended to a free basis for R^n, since $R^n = P \oplus Rv$. Since $P \cong R^n/Rv$, P will be free if and only if it has $n-1$ generators, and so P will be free if only if v can be extended to a free basis for R^n. This led to the following question: if one is given one column of a matrix consisting of polynomials over K that generate the unit ideal, can one "complete" the matrix so that it has determinant which is a unit in the polynomial ring? This is equivalent to completing the matrix so that its determinant is 1 if $n \geq 2$: the unit can be absorbed into one of the columns other than the first. This is known as the "unimodular column" problem. However, some authors, who use matrices that act on the right, study the equivalent "unimodular row" problem.

The question was raised by Serre in the mid 1950s and was open until 1976, when it was settled in the affirmative, independently, by D. Quillen and A. Suslin. A bit later, Vaserstein gave another proof which is very short, albeit very tricky. It is true that projective modules over a polynomial ring over a field are free, but it is certainly a non-trivial theorem.