The Functor Tor

Basic Properties of Tor

Let R be a commutative ring. The functors $\mathrm{Tor}_i^R(A,\, B)$ are functors of two variable R-modules A and B that are covariant in each module when the other is held fixed. This is similar to the behavior of $A \otimes_R B$: in fact, $\mathrm{Tor}_0^R(A,\, B) = A \otimes_R B$. The superscript R may be omitted when the ring R is clear from context. The Tor functors are introduced because tensor product, in general, does not preserve injectivity of maps. The following are the basic properties of Tor. For the purpose of this course, it is more than sufficient to know these.

(1) $\mathrm{Tor}_0^R(A, B) \cong A \otimes_R B$ as functors of two variables.

(2) $\mathrm{Tor}_i(A, B) = 0$ is $i < 0$.

(3) If A is projective (or flat), $\mathrm{Tor}_i(A,\, B) = 0$ if $i > 0$. In particular, this holds when A is free.

For the purpose of stating the next fact it is convenient to discuss sequences of modules and maps indexed by $\mathbb{Z}$. Such a sequence has the form

$$\ldots \to M_{i+1} \xrightarrow{d_{i+1}} M_i \xrightarrow{d_i} M_{i-1} \to \ldots$$

or

$$\ldots \to M_{i-1} \xrightarrow{d_{i-1}} M_i \xrightarrow{d_i} M_{i+1} \to \ldots\,,,$$

that is, the maps may lower degrees by one or increase degrees by one. Such a sequence is called a *complex* if the composition of any two consecutive maps is 0 and is called *exact* if the image of each map is the same as the kernel of the next. By a morphism of sequences from $\cdots \to M_i \to \cdots$ to $\cdots N_i \to \cdots$ we mean a collection of maps $\phi_i : M_i \to N_i$ such that the diagrams

$$\begin{array}{ccc} M_i & \longrightarrow & M_{i\pm 1} \\ {\scriptstyle \phi_i}\big\downarrow & & {\scriptstyle \phi_{i\pm 1}}\big\downarrow \\ N_i & \longrightarrow & N_{i\pm 1} \end{array}$$

all commute. Thus, sequences form a category, and complexes form a full subcategory, Likewise, exact sequences form a subcategory. A *short exact sequence* is one where all but three consecutive terms are 0, and we may assume that the possibly nonzero terms occur at the spots indexed 0, 1, 2 for the purpose of defining morphisms, so that we have a notion of morphisms of short exact sequences.

(4) If $0 \to A \to B \to C \to 0$ is an exact sequence of R-modules and M is an R-module there is a (typically infinite) long exact sequence

$$\cdots \to \mathrm{Tor}_i(A, M) \to \mathrm{Tor}_i(B, M) \to \mathrm{Tor}_i(C, M) \to \mathrm{Tor}_{i-1}(A, M) \to \cdots$$
$$\to \mathrm{Tor}_1(A, M) \to \mathrm{Tor}_1(B, M) \to \mathrm{Tor}_1(C, M) \to A \otimes_R M \to B \otimes_R M \to C \otimes_R M \to 0.$$

This exact sequence is covariantly functorial in the short exact sequence $0 \to A \to B \to C \to 0$ when M is held fixed and is covariantly functorial in N when the $0 \to A \to B \to C \to 0$ is held fixed. The maps are the ones given by the functoriality of Tor_i^R in each input module when the other is held fixed.

(5) $\mathrm{Tor}_i(A, B) \cong \mathrm{Tor}_i(B, A)$ as functors of two variables.

(6) The map induced by multiplication x on A (or on B) is multiplication by x on $\mathrm{Tor}_i^R(A,\, B)$.

(7) The module $Tor_i^R(A,\, B)$ is killed by $\mathrm{Ann}_R A + \mathrm{Ann}_R B$.

(8) Tor commutes with arbitrary direct sum and arbitrary colimits (i.e., direct limits).

(9) If R is Noetherian and M, N are finitely generated, all the modules $\mathrm{Tor}_i(M, N)$ are finitely generated.

(10) If M is flat over R, $\mathrm{Tor}_i^R(M \otimes A,\, B) \cong M \otimes_R \mathrm{Tor}_i^R(A,\, B)$.

(11) If S is a flat R-algebra, $S \otimes_R \mathrm{Tor}_i^R(A, B) \cong \mathrm{Tor}_i^S(S \otimes_R A, S \otimes_R B)$.

(12) If $0 \to A' \to F \to A \to 0$ is exact, then $\mathrm{Tor}_1^R(A, B)$ is the kernel of $A' \otimes_R B \to F \otimes_R B$ and $\mathrm{Tor}_{i+1}^R(A, B) \cong \mathrm{Tor}_i(A', B)$ for all $i \geq 1$.

(13) If I, J are ideals of R, $\mathrm{Tor}_1^R(R/I, R/J) \cong (I \cap J)/(IJ)$.

The construction of these functors and the proofs of these properties are given in the next section. We note here that the functors Tor_i are uniquely determined, up to canonical isomorphism, by the first four properties, that (7) follows easily from (6), that (12) follows from (1) — (4) and that (13) follows from (12).

Construction of Tor

In order to develop the theory of Tor, for which we need to talk about projective resolutions. Let R be any ring, and M be any R-module. Then it is possible to map a projective R-module P onto M. In fact one can choose a set of generators $\{u_\lambda\}_{\lambda \in \Lambda}$ for M, and then map the free module $P = \bigoplus_{\lambda \in \Lambda} R b_\lambda$ on a correspondingly indexed set of generators $\{b_\lambda\}_{\lambda \in \Lambda}$ onto M: there is a unique R-linear map $P \twoheadrightarrow M$ that sends $b_\lambda \to u_\lambda$ for all $\lambda \in \Lambda$. Whenever we have such a surjection, the kernel M' of $P \twoheadrightarrow M$ is referred to as a *first module of syzygies* of M. We define k th modules of syzygies by recursion: a k th module of syzygies of a first module of syzygies is referred to as a $k+1$ st module of syzygies.

There is even a completely canonical way to map a free module onto M. Given M let $\mathcal{F}(M)$ denote the module of all functions from M to R that vanish on all but finitely many elements of M. This module is R-free on a basis $\{b_m\}_{m \in M}$ where b_m is the function that is 1 on m and 0 elsewhere. The map that sends $f \in \mathcal{F}(M)$ to $\sum_{m \in M} f(m)m$ is a canonical surjection: note that it maps b_m to m. The sum makes sense because all but finitely many terms are 0.

By a *projective resolution* of M we mean an infinite sequence of projective modules

$$\cdots \to P_n \to \cdots \to \; P_1 \to P_0 \to 0$$

which is exact at P_i for $i > 0$, together with an isomorphism $P_0/\mathrm{Im}\,(P_1) \cong M$. Recall the exactness at P_i means that the image of the map into P_i is the kernel of the map from P_i. Note that it is equivalent to give an exact sequence

$$\cdots \to P_n \to \cdots \to \; P_1 \to P_0 \twoheadrightarrow M \to 0$$

which is exact everywhere. A projective resolution is called *finite* if $P_n = 0$ for all sufficiently large n.

We can always construct a projective resolution of M as follows: map a projective module P_0 onto M. Let Z_1 be the kernel, a first module of syzygies of M. Map a projective module P_1 onto Z_1. It follows that $P_1 \to P_0 \to M \to 0$ is exact, and Z_2, the kernel of $P_1 \to P_0$, is a second module of syzygies of M. Proceed recursively. If $P_n \to \cdots \to P_1 \to P_0 \to M \to 0$ has been constructed so that it is exact (except at P_n), let Z_n be the kernel of $P_n \to P_{n-1}$), which will be an n th module of syzygies of M. Simply map a projective P_{n+1} onto Z_n, and use the composite map

$$P_{n+1} \twoheadrightarrow Z_n \subseteq P_n$$

to extend the resolution.

One can form a completely canonical resolution that is free, not merely projective, by taking $P_0 = \mathcal{F}(M)$ together with the canonical map $\mathcal{F}(M) \twoheadrightarrow M$ to begin, and choosing $P_{n+1} = \mathcal{F}(Z_n)$ along with the canonical map $\mathcal{F}(Z_n) \to Z_n$ at the recursive step. We refer to this as the *canonical* free resolution of M. We shall see that one can compute Tor using any projective resolution, but it is convenient for the purpose of having an unambiguous definition at the start to have a canonical choice of resolution.

If M is an R-module, we define $\mathrm{Tor}_n^R(M,\,N)$ to be the n th homology module of the complex $\cdots \to P_n \otimes_R N \to \cdots \to P_1 \otimes_R N \to P_0 \otimes_R N \to 0$, i.e., $H_n(P_\bullet \otimes_R N)$, where $P_\bullet$ is the canonical free resolution of M. The n th homology module of a complex $G_\bullet$ is Z_n/B_n where Z_n is the kernel of the map $G_n \to G_{n-1}$ and B_n is the image of the map $G_{n+1} \to G_n$.

Despite the unwieldy definition, the values of $\mathrm{Tor}^R(M,\,N)$ are highly computable. One might take the view that all of the values of Tor make a small correction for the fact that tensor is not an exact functor. The values of Tor are not always small, but one can often show that Tor vanishes, or has finite length, and the information it can provide is very useful.

We make some conventions that will be useful in dealing with complexes.

By a *sequence* of R-modules (and maps, although they will usually not be mentioned) we mean a family of modules $\{M_n\}_{n\in\mathbb{Z}}$ indexed by the integers, and for every $n \in \mathbb{Z}$ an R-linear map $d_n : M_n \to M_{n-1}$. (We restrict here to the case where the maps lower degrees by one: the case where the maps raise degrees by one is treated by renumbering.) The sequence is called a *complex* if $d_n \circ d_{n+1} = 0$ for all $n \in \mathbb{Z}$. This is equivalent to the condition that $\mathrm{Im}\,(d_{n+1}) \subseteq \mathrm{Ker}\,(d_n)$ for all n. We often use the notation $M_\bullet$ to denote a complex of modules. We define $H_n(M_\bullet)$ to be $\mathrm{Ker}\,(d_n)/\mathrm{Im}\,(d_{n+1})$, the n th *homology* module of $M_\bullet$. We shall make the homology modules into a new complex, somewhat artificially, by defining all the maps to be 0. Given a complex $M_\bullet$ we make the convention $M^n = M_{-n}$ for all $n \in \mathbb{Z}$. Thus, the same complex may be indicated either as

$$\cdots \to M_{n+1} \to M_n \to M_{n-1} \to \cdots \to M_1 \to M_0 \to M_{-1} \to$$

$$\cdots \to M_{-(n-1)} \to M_{-n} \to M_{-(n+1)} \to \cdots$$

or as

$$\cdots \to M^{-(n+1)} \to M^{-n} \to M^{-(n-1)} \to \cdots \to M^{-1} \to M^0 \to M^1 \to$$

$$\cdots \to M^{n-1} \to M^n \to M^{n+1} \to \cdots$$

for which we write $M^\bullet$. With these conventions, $H^i(M^\bullet) = H_{-i}(M_\bullet)$. Thus, there really isn't any distinction between cohomology ($H^i(M^\bullet)$) and homology. A complex that is exact at every spot is called an *exact* sequence.

By a morphism of sequences $M_\bullet \to M'_\bullet$ we mean a family of R-linear maps $\phi_n : M_n \to M'_n$ such that for every $n \in \mathbb{Z}$ the diagram

$$\begin{tikzcd} M_n \arrow[r, "d_n"] \arrow[d, "\phi_n"'] & M_{n-1} \arrow[d, "\phi_{n-1}"] \\ M'_n \arrow[r, "d'_n"'] & M'_{n-1} \end{tikzcd}$$

commutes. There is an obvious notion of composition of morphisms of sequences: if $\phi : M_\bullet \to M'_\bullet$ and $\psi : M'_\bullet \to M''_\bullet$, let $\psi\circ\phi : M_\bullet \to M''_\bullet$ be such that $(\psi\circ\phi)_n = \psi_n\circ\phi_n$. Then sequences of R-modules and morphisms is a category (the identity map from $M_\bullet \to M_\bullet$ is, in degree n, the identity map $M_n \to M_n$).

Given a category $\mathcal{C}$, we say that $\mathcal{D}$ is a *full subcategory* of $\mathcal{C}$ if $\mathrm{Ob}\,(\mathcal{D}) \subseteq \mathrm{Ob}\,(\mathcal{C})$ and for all objects X and Y of $\mathcal{D}$, $\mathrm{Mor}_{\mathcal{D}}(X, Y) = \mathrm{Mor}_{\mathcal{C}}(X, Y)$. Composition in $\mathcal{D}$ is the same as composition in $\mathcal{C}$, when it is defined. Note that for every subclass of $\mathrm{Ob}\,(\mathcal{C})$ there is a unique full subcategory of $\mathcal{C}$ with these as its objects. For example, finite sets and functions is a full subcategory of sets and functions, abelian groups and group homomorphisms is a full subcategory of groups and group homomorphisms, and Hausdorff topological spaces and continuous maps is a full subcategory of topological spaces and maps.

The category of complexes of R-modules is defined as the full subcategory of the category of sequences of R-modules whose objects are the complexes of R-modules. We define a *left* complex $M_\bullet$ as a complex such that $M_n = 0$ for all $n < 0$, and a *right complex* as a complex such that $M_n = 0$ for all $n > 0$. Thus, a left complex has the form

$$\cdots \to M_n \to M_{n-1} \to \cdots \to M_1 \to M_0 \to 0 \to 0 \to \cdots$$

and a right complex has the form

$$\cdots \to 0 \to 0 \to M_0 \to M_{-1} \to \cdots \to M_{-(n-1)} \to M_{-n} \to \cdots$$

which we may also write, given our conventions, as

$$\cdots \to 0 \to 0 \to M^0 \to M^1 \to \cdots \to M^{n-1} \to M^n \to \cdots$$

Left complexes and right complexes are also full subcategories of sequences (and of complexes).

A complex is called *projective* (respectively, *free*) if all of the modules occurring are projective (respectively, free).

By a *short exact sequence* we mean an exact sequence of modules $M_\bullet$ such that $M_n = 0$ except possibly when $n \in \{0,\, 1,\, 2\}$:

$$0 \to M_2 \to M_1 \to M_0 \to 0.$$

These also form a full subcategory of complexes. The numbering is not very important here. We shall also refer to M_2 as the *leftmost* module, M_1 as the *middle* module, and M_0 as the *rightmost* module in such a sequence.

The homology modules of a complex may be regarded as a complex by taking all the maps to be 0. The homology operator is then in fact a covariant functor from complexes to complexes: given a map $\{\phi_n\}_n$ of complexes $M_\bullet \to M'_\bullet$, with maps $\{d_n\}_n$ and $\{d'_n\}_n$ respectively, note that if $d_n(u) = 0$, then

$$d'_n\bigl(\phi_n(u)\bigr) = \phi_{n-1}\bigl(d_n(u)\bigr) = \phi_{n-1}(0) = 0,$$

so that ϕ maps $\mathrm{Ker}\,(d_n)$ into $\mathrm{Ker}\,(d'_n)$. If $u = d_{n+1}(v)$, then

$$\phi_n(u) = \phi_n\bigl(d_{n+1}(v)\bigr) = d'_{n+1}\bigl(\phi_{n+1}(v)\bigr),$$

which shows that ϕ_n maps $\mathrm{Im}\,(d_{n+1})$ into $\mathrm{Im}\,(d'_{n+1})$. This implies that ϕ_n induces a map of homology

$$H_n(M_\bullet) = \mathrm{Ker}\,(d_n)/\mathrm{Im}\,(d_{n+1}) \to \mathrm{Ker}\,(d'_n)/\mathrm{Im}\,(d'_{n+1}) = H_n(M'_\bullet).$$

This is easily checked to be a covariant functor from complexes to complexes.

In this language, we define a *projective resolution* of an R-module M to be a left projective complex $P_\bullet$ such that $H_n(P_\bullet) = 0$ for $n \geq 1$ together with an isomorphism $H_0(P_\bullet) \cong M$. Since $H_0(P_\bullet) \cong P_0/\mathrm{Im}\,(P_1)$, giving an isomorphism $H_0(P_\bullet) \cong M$ is equivalent to giving a surjection $P_0 \twoheadrightarrow M$ whose kernel is $\mathrm{Im}\,(P_1)$. Thus, giving a projective resolution of M in the sense just described is equivalent to giving a complex

$$(*) \qquad \cdots \to P_n \to \cdots \to P_1 \to P_0 \twoheadrightarrow M \to 0$$

that is exact, and such that P_n is projective for $n \geq 0$. In this context it will be convenient to write $P_{-1} = M$, but it must be remembered that P_{-1} need not be projective. The complex $(*)$ will be referred to as an *augmented projective resolution* of M.

We recall that an R-module P is projective if and if, equivalently

(1) When $M \twoheadrightarrow N$ is onto, $\mathrm{Hom}_R(P,\, M) \to \mathrm{Hom}_R(P,\, N)$ is onto.

(2) $\mathrm{Hom}_R(P,\, \_\,)$ is an exact functor.

(3) P is a direct summand of a free module.

A direct sum of modules (finite or infinite) is projective if and only if all of the summands are. It is easy to verify (1) for free modules: if P is free on the free basis $\{b_\lambda\}_{\lambda\in\Lambda}$ and $M \twoheadrightarrow N$ is onto, given a map $f : P \to N$, we lift to a map $g : P \to M$ as follows: for each free basis element b_λ of P, choose $u_\lambda \in M$ that maps to $f(b_\lambda)$, and let $g(b_\lambda) = u_\lambda$.

We next want to define what it means for two maps of complexes of R-modules to be homotopic. Let $P_\bullet$ and $N_\bullet$ be two complexes. First note that the set of maps of complexes $\mathrm{Mor}\,(P_\bullet,\; N_\bullet)$ is an R-module: we let

$$\{\phi_n\}_n + \{\psi_n\}_n = \{\phi_n + \psi_n\}_n,$$

and

$$r\{\phi_n\}_n = \{r\phi_n\}_n.$$

We define $\{\phi_n\}_n$ to be *null homotopic* or *homotopic* to 0 if there exist maps $h_n : P_n \to N_{n+1}$ (these are *not* assumed to commute with the complex maps) such that for all n,

$$\phi_n = d'_{n+1}h_n + h_{n-1}d_n.$$

The set of null homotopic maps is an R-submodule of the R-module of maps of complexes. Note that the homology functor $H_\bullet$ is R-linear on maps of complexes.

Two maps of complexes are called *homotopic* if their difference is null homotopic.

Lemma. *If two maps of complexes are homotopic, they induce the same map of homology.*

Proof. We have
$$\phi_n - \phi'_n = d'_{n+1}h_n + h_{n-1}d_n$$
for all n. Let $z \in \mathrm{Ker}\,(d_n)$. Then
$$\phi_n(z) - \phi'_n(z) = d'_{n+1}\big(h_n(z)\big) + h_{n-1}\big(d_n(z)\big).$$
The second term is 0, since $d_n(z) = 0$, and the first term is in $\mathrm{Im}\,(d'_{n+1})$. This shows that
$$[\phi_n(z)] - [\phi'_n(z)] = 0,$$
as required. □

The following Theorem is critical in developing the theory of derived functors such as Tor and Ext. In the applications a will typically be 0, but the starting point really does not matter.

Theorem. *Let $P_\bullet$ and $N_\bullet$ be complexes such that $P_n = 0$ for $n < a-1$ and $N_n = 0$ for $n < a-1$. Suppose that $N_\bullet$ is exact, and that P_n is projective for $n \geq a$. Let $M = P_{a-1}$ (which need not be projective) and $N = N_{a-1}$. Let ϕ be a given R-linear map from M to N. Then we can choose $\phi_n : P_n \to N_n$ for all $n \geq a$ such that, with $\phi_{a-1} = \phi$, $\{\phi_n\}_n$ is a map of complexes (of course, $\phi_n = 0$ is forced for $n < a-1$). Briefly, ϕ lifts to a map $\{\phi_n\}_n$ of complexes. Moreover, any two different choices $\{\phi_n\}_n$ and $\{\phi'_n\}_n$ for the lifting (but with $\phi_{a-1} = \phi'_{a-1} = \phi$) are homotopic.*

Proof of existence. We have a composite map $P_a \to M \to N$ and a surjection $N_a \twoheadrightarrow N$. Therefore, by the universal mapping property of projective modules, we can choose an R-linear map $\phi_a : P_a \to N_a$ such that $\phi \circ d_a = d'_a \circ \phi_a$. We now shorten both complexes: we replace the right end
$$N_{a+1} \to N_a \twoheadrightarrow N \to 0$$
of $N_\bullet$ by
$$N_{a+1} \to N' \to 0,$$
where N' is the image of N_{a+1} in N_a, which is also $\mathrm{Ker}\,(N_a \to N)$. We shorten the complex $P_\bullet$ by replacing the right end
$$P_{a+1} \to P_a \to M \to 0$$
by
$$P_{a+1} \to M' \to 0,$$
where M' is the kernel of $P_a \to M$. The restriction of ϕ_a to M' gives a map ϕ' of M' to N'. We are now in precisely the same situation that we started with, and we construct ϕ_{a+1} in the same manner that we constructed ϕ_a. The existence of all the ϕ_n follows by a straightforward induction. □

Proof of uniqueness up to homotopy. We work with the difference of the two liftings. It therefore suffices to show that a lifting of the 0 map $M \to N$ is null homotopic. Of course, we must define $h_n = 0$ if $n < a-1$, and we define $h_{a-1} = 0$ as well: the property we need holds because $\phi = 0$. We construct the maps h_n recursively. Suppose that we have constructed h_n for $n < b$ where $b \geq a$ such that

$$\phi_n = d'_{n+1}h_n + h_{n-1}d_n$$

for all $n < b$. It will suffice to construct $h_b : P_b \to N_{b+1}$ such that

$$\phi_b = d'_{b+1}h_b + h_{b-1}d_b.$$

We claim that the image of $\phi_b - h_{b-1}d_b$ is contained in the image of N_{b+1}. By the exactness of $N_\bullet$, it suffices to show that the image of $\phi_b - h_{b-1}d_b$ is contained in the kernel of d'_b, i.e.,

$$d'_b\phi_b - d'_bh_{b-1}d_b = 0.$$

But since

$$\phi_{b-1} = d'_bh_{b-1} + +h_{b-2}d_{b-1},$$

we may substitute

$$d'_bh_{b-1} = \phi_{b-1} - h_{b-2}d_{b-1}$$

to get

$$d'_b\phi_b - (\phi_{b-1} - h_{b-2}d_{b-1})d_b.$$

since $d_{b-1}d_b = 0$, this is just

$$d'_b\phi_b - \phi_{b-1}d_b = 0$$

since $\{\phi_n\}_n$ is a map of complexes. Since

$$\alpha = \phi_b - h_{b-1}d_b$$

has image in $\mathrm{Im}\,(N_{b+1})$, we may let β be α with its target restricted to $\mathrm{Im}\,(N_{b+1})$. Since P_b is projective and d'_{b+1} maps onto the target of β, we may lift β to a map $h_b : P_b \to N_{b+1}$, so that $d'_{b+1}h_b = \beta$, which implies that

$$d'_{b+1}h_b = \phi_b - h_{b-1}d_b,$$

as required. $\square$

Remark. Consider the case where $a = 0$. We also have maps of complexes once the augmentations $P_{-1} = M$ and $N_{-1} = N$ are dropped, and because $h_{-1} = 0$, we still have homotopic maps of complexes.

The significance of the result just proved is that we can use *any* projective resolution of M to calculate Tor — up to canonical isomorphism.

Theorem. *Let $P_\bullet$ and $Q_\bullet$ be projective resolutions of the R-module M. Choose a lifting of* id_M *to a map of resolutions $\phi_\bullet : P_\bullet \to Q_\bullet$ and also to a map of resolutions $\psi_\bullet : Q_\bullet \to P_\bullet$. Then $\phi_\bullet \otimes_R \mathrm{id}_N$ and $\psi_\bullet$ induce mutually inverse isomorphisms between $H_\bullet(P_\bullet \otimes_R N)$ $H_\bullet(Q_\bullet \otimes_R N)$ that are independent of the choices of the ϕ and ψ. In this sense, any projective resolution of M may be used to compute all the modules $\mathrm{Tor}_n^R(M, N)$ up to canonical isomorphism.*

Proof. If we took a different choice of $\phi_\bullet$ it would be homotopic to the original. The homotopy is preserved when we apply $\_ \otimes_R N$. Therefore we get maps of homology that are independent of the choice of $\phi_\bullet$. The same remark applies to $\psi_\bullet$. The composition $\psi_\bullet \circ \phi_\bullet$ gives a map of complexes $P_\bullet \to P_\bullet$ that lifts id_M. The identity map of complexes is also such a lifting. This shows that $\psi \circ \phi$ is homotopic to the identity map on $P_\bullet$. This homotopy is preserved when we apply $\_ \otimes_R N$. This shows that the composition of the induced maps of homology is the identity map. The argument is the same when the composition is taken in the other order. $\square$

Notice that $Tor_n^R(M, N) = 0$ if $n < 0$. If

$$\cdots \to P_1 \to P_0 \twoheadrightarrow M \to 0$$

is a projective resolution of M, then

$$\mathrm{Tor}_0^R(M, N) = H_0(\cdots \to P_1 \otimes_R N \to P_0 \otimes_R N \to 0) \cong \frac{P_0 \otimes_R N}{\mathrm{Im}\,(P_1 \otimes_R N)} \cong \frac{P_0}{\mathrm{Im}\,(P_1)} \otimes N$$

using the right exactness of tensor. Since

$$\frac{P_0}{\mathrm{Im}\,(P_1)} \cong M,$$

we have that

$$\mathrm{Tor}_0^R(M, N) \cong M \otimes N.$$

We now give an alternative point of view about complexes. Let $R[d] = R[\Delta]/\Delta^2$, and give Δ degree -1. The category of sequences is the same as the category of $\mathbb{Z}$-graded $R[\Delta]$-modules and degree preserving maps. The category of complexes is the same as the full subcategory of $\mathbb{Z}$-graded $R[d]$-modules and degree-preserving maps. It is very easy to see that given $M_\bullet \to M'_\bullet$, one has induced maps $\mathrm{Ann}_{M_\bullet} d \to \mathrm{Ann}_{M'_\bullet} d$ and $dM_\bullet \to dM'_\bullet$. Homology is recovered as $\mathrm{Ann}_{M_\bullet} d/dM_\bullet$, This is an $R[d]$-module on which d acts trivially, and it is now quite obvious that there are induced maps $H_\bullet(M_\bullet) \to H_\bullet(M'_\bullet)$ of homology.

From this point of view, the map h that gives a null homotopy is a degree 1 map of graded R-modules, that is, it increases degrees of homogeneous elements by 1: it need not commute with d. Then $hd + dh$ preserves degree, and does commute with d:

$$d(hd + dh) = dhd = (hd + dh)d.$$

$hd+dh$ gives the zero map on homology because if $dz = 0$, $(hd+dh)(z) = d(h(z)) \in \mathrm{Im}\,(d)$.

We next want to show that Tor is a covariant functor of two variables. Given an R-module map $M \to M'$ it lifts to a map of projective resolutions $P_\bullet$ for M and $P'_\bullet$ for M'. This gives induced maps of homology when we apply $\_ \otimes N$. If we choose a different lifting we get homotopic maps of complexes and the homotopy is preserved when we apply $\_ \otimes_R N$. The check of functoriality in M is straightforward.

Given a map $N \to N'$, we get obvious induced maps $P_\bullet \otimes N \to P_\bullet \otimes N'$ that yield the maps of Tor. Once again, the proof of functoriality is straightforward.

In order to develop the theory of Tor further, we want to consider double complexes. One point of view is that a double complex consists of a family of R-modules $\{M_{ij}\}_{i,j\in\mathbb{Z}}$ together with "horizontal" R-module maps $d_{ij} : M_{ij} \to M_{i,j-1}$ and "vertical" R-module maps $d'_{ij} : M_{ij} \to M_{i-1,j}$ for all $i, j \in \mathbb{Z}$, such that every $d_{ij}d_{i,j+1} = 0$ (the rows are complexes), every $d'_{i,j}d'_{i+1,j} = 0$ (the columns are complexes) and such that all of the squares

$$\begin{array}{ccc} M_{ij} & \xrightarrow{d_{i,j}} & M_{i,j-1} \\ {\scriptstyle d'_{ij}}\big\downarrow & & \big\downarrow{\scriptstyle d'_{i,j-1}} \\ M_{i-1,j} & \xrightarrow[d_{i-1,j}]{} & M_{i-1,j-1} \end{array}$$

commute: omitting subscripts, this means that $d'd = dd'$. An alternative convention that is sometimes made instead is that in a double complex, the vertical and horizontal differentials anticommute: i.e., $d'd = -dd'$. Both conventions have advantages and disadvantages: we shall call the latter type of double complex a *signed double complex*, but this terminology is not standard.

Given a double complex in our sense, one can alway create a signed double complex by altering the signs on some of the maps. To have a standard way of doing this, our convention will be that the associated signed double complex is obtained by replacing d'_{ij} by $(-1)^i d'_{ij}$, while not changing any of the d_{ij}. There are many ways to alter signs to get the squares to anticommute. It does not matter which one is used in the sense that the homology of the total complex (we shall define the total complex momentarily) is unaffected.

An alternative point of view is obtained by working with $\bigoplus_{ij} M_{ij}$, a $(\mathbb{Z}\times\mathbb{Z})$-graded R-module. Let Δ and Δ' be indeterminates over R, and let $R[d,d'] = R[\Delta,\Delta']/(\Delta^2,\Delta'^2)$, where Δ has degree $(0,-1)$, Δ' has degree $(-1,0)$, and d, d' are their images. The d_{ij} define an action of d on $\bigoplus_{ij} M_{ij}$ that lowers the second index by 1, and the d'_{ij} define an action of d' on $\bigoplus_{ij} M_{ij}$ that lowers the first index by 1. Thus, a double complex is simply a $(\mathbb{Z}\times\mathbb{Z})$-graded $R[d,d']$-module.

A signed double complex may be thought of as a $(\mathbb{Z}\times\mathbb{Z})$-graded module over the noncommutative ring Λ generated over R by elements d and d' of degrees $(0,-1)$ and

$(-1, 0)$, respectively, satisfying $d^2 = {d'}^2 = 0$ and $dd' = -d'd$. Λ may be identified with the exterior algebra over R of the free R-module $Rd \oplus Rd'$.

A *morphism* of double complexes is a bidegree-preserving $\mathbb{Z} \times \mathbb{Z}$-graded $R[d, d']$-module homomorphism, so that the maps commute with the actions of d and of d'. We indicate a double complex, whether signed or not, with the notation $M_{\bullet\bullet}$: the subscript is a reminder that the bidegree has two integer components. The *total complex* of a signed double complex $M_{\bullet\bullet}$, denoted $\mathcal{T}_\bullet(M_{\bullet\bullet})$, is obtained by letting $\mathcal{T}_n(M_{\bullet\bullet}) = \bigoplus_{i+j=n} M_{ij}$, with differential $d+d'$. This is indeed a complex because $(d+d')(d+d') = d^2+d'd+dd'+{d'}^2 = 0$. The *total complex* of a double complex $M_{\bullet\bullet}$ is simply the total complex of the associated signed double complex. This means that the differential, restricted to M_{ij}, is $d_{ij}+(-1)^i d'_{ij}$.

Example. If $M_\bullet$ and $N_\bullet$ are complexes with differentials $d_\bullet$ and $d'_\bullet$, respectively, we get a double complex $M_\bullet \otimes N_\bullet$ whose i, j term is $M_j \otimes N_i$. Thus, the i th row is

$$\cdots \to M_{j+1} \otimes_R N_i \to M_j \otimes_R N_i \otimes_R M_{j-1} \otimes_R N_i \to \cdots$$

and the j th column is

$$\begin{array}{c} \vdots \\ \downarrow \\ M_j \otimes_R N_{i+1} \\ \downarrow \\ M_j \otimes_R N_i \\ \downarrow \\ M_j \otimes_R N_{i-1} \\ \downarrow \\ \vdots \end{array}$$

The differentials in the i th row are the maps $d_j \otimes \mathrm{id}_{N_i}$ while those in the j th column are the maps $\mathrm{id}_{M_j} \otimes d'_i$. We shall return to the study of double complexes of this form shortly. The total complex $\mathcal{T}_\bullet(M_\bullet \otimes_R N_\bullet)$ is called the *total tensor product* of $M_\bullet$ and $N_\bullet$, and some authors omit the word "total," but we reserve the term "tensor product" for the double complex. Note that the differential of the total tensor product applied to $u_j \times v_i$ has the value $du_j \otimes v_i + (-1)^j u_j \otimes d'v_i$.

Given a double complex, one can take homology first of the rows (giving a new double complex) and then of the columns. The result is called *iterated* homology. One can also take homology first of the columns and then of the rows: this gives the iterated homology for the other order. Third, one can take homology of the total complex. These three objects are related in a complicated way. One of the most important applications of the theory of spectral sequences is to explain the relationship. We shall return to these ideas later.

For the moment, we want to prove two lemmas about double complexes that are of immense importance. They are both special cases of the theory of spectral sequences, but we ignore this for the moment.

The first is the *snake* or *serpent* lemma. One starts with a short exact sequence of complexes

$$0 \to A_\bullet \xrightarrow{\alpha} B_\bullet \xrightarrow{\beta} C_\bullet \to 0,$$

which simply means that for all n, the sequence $0 \to A_n \to B_n \to C_n \to 0$ is exact. We may form from these a double complex in which $A_\bullet$, $B_\bullet$ and $C_\bullet$ are the columns. A typical row is then $0 \to A_n \to B_n \to C_n \to 0$, and so is exact. A key point is that in this situation there is a well-defined map $\gamma_\bullet$ from $H_\bullet(C_\bullet) \to H_{\bullet-1}(A_\bullet)$ called *the connecting homomorphism*, where the subscript $\,_{\bullet-1}$ indicates that degrees have been shifted by -1, so that the $\gamma_n : H_n(A_\bullet) \to H_{n-1}(C_\bullet)$. We could also have used our graded module conventions and written $H_\bullet(C_\bullet)(-1)$, but we shall use the other convention for shifting the numbering of complexes.

The definition of γ is quite simple: since every map $B_n \to A_n$ is onto, given a cycle $z \in A_n$ we may choose $b \in B_n$ such that $\beta(b) = z$. Since z maps to 0 in A_{n-1}, we have that $\beta(db) = d\big(\beta(b)\big) = dz = 0$ maps to 0 in B_{n-1}, and so db is the image of a unique element $a \in A_{n-1}$. Moreover $da = 0$, since $d\big(\alpha(a)\big) = d(db) = 0$. Our map will send $[z] \in H_n(C_\bullet)$ to $[a] \in H_{n-1}(A_\bullet)$. Note that if had made another choice of b mapping to z, it would have the form $b + \alpha(a_1)$ for some $a_1 \in A_n$. Then $d(b + \alpha(a_1)) = db + \alpha(da_1)$, and a would change to $a + d(a_1)$, which does not change its homology class. If we change the choice of representative z to $z + dc'$ for some $c' \in C_{n+1}$, we can choose $b' \in B_{n+1}$ that maps to c', and then a new choice for b is $b + db'$. But $d(b + db') = db$. This shows that we have a well-defined map $H_n(C) \to H_{n-1}(A)$. R-linearity follows from the fact that if b_1 and b_2 map to z_1 and z_2, then $rb_1 + b_2$ maps to $rz_1 + z_2$ for $r \in R$. Very briefly, the connecting homomorphism is characterized by the formula $\gamma([\beta(b)] = [\alpha^{-1}(db)]$, which makes sense since α is injective and db is in its image when $\beta(b)$ is a cycle.

Note the following picture:

$$\begin{array}{ccccc} & & b & \mapsto & z \\ & & \downarrow & & \\ a & \mapsto & db & & \\ \downarrow & & & & \\ 0 & & & & \end{array}$$

Proposition (snake or serpent lemma). *If $0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$ is a short exact sequence of complexes, then there is a long exact sequence of homology:*

$$\cdots \to H_{n+1}(C_\bullet) \xrightarrow{\gamma_{n+1}} H_n(A_\bullet) \xrightarrow{\alpha_{n*}} H_n(B_\bullet) \xrightarrow{\beta_{n*}} H_n(C_\bullet) \xrightarrow{\gamma_n} H_{n-1}(A_\bullet) \to \cdots$$

where α_{n*} and β_{n*} are the maps of homology induced by α_n and β_n, respectively.

Moreover, given a morphism of short exact sequences of complexes (this makes sense, thinking of them as double complexes), we get an induced morphism of long exact sequences, and the construction is functorial.

Proof. It suffices to check exactness at $H_n(C_\bullet)$, $H_n(B_\bullet)$, and $H_n(A_\bullet)$.

A cycle z in C_n is killed by γ iff for b mapping to c, db is the image of $a \in A_{n-1}$ that is a boundary, i.e., that has the form da' for some $a' \in A_{n-1}$. But then $b - a'$ is a cycle in B_n that maps to z, which shows that $[b - a']$ maps to $[z]$, as required. Conversely, if b is a cycle that maps to z, $db = 0$ and it is immediate that $[z]$ is in the kernel of γ_n.

For a cycle in $z \in B_n$, $[z]$ is killed by β_{n*} iff $\beta(z)$ is a boundary in C_n, i.e., $\beta(z) = dc'$, where $c' \in C_{n+1}$. Choose $b' \in B_{n+1}$ that maps onto c'. Then $z - db'$ maps to 0 in C_n, and so is the image of an element $a \in A_n$: moreover, da maps to $dz - d^2b' = 0 - 0$, and $A_{n-1} \hookrightarrow B_{n-1}$, so that a is cycle and $[a]$ maps to $[z]$. Conversely, the fact that the composite $H_n(A_\bullet) \to H_n(B_\bullet) \to H_n(C_\bullet)$ is 0 is immediate from the fact that $\beta\alpha = 0$.

Finally, let $z \in A_n$ be a cycle such that $[z]$ is zero in $H_n(B_\bullet)$. Then $\alpha(z)$ is a boundary, i.e., $\alpha(z) = db$ for $b \in B_{n+1}$. By the definition of γ_{n+1} we have that $\gamma_{n+1}([\beta(b)]) = [a]$. Conversely, if $\gamma_{n+1}([\beta(b)]) = [a]$ we have that $[a]$ maps to $[db] = 0$, so that $\alpha_{n*}\gamma_{n+1} = 0$.

Suppose that one has a morphism of short exact sequences from

$$0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$$

to

$$0 \to A'_\bullet \to B'_\bullet \to C'_\bullet \to 0.$$

The functoriality of the long exact sequence is immediate from the functoriality of taking homology, except for the commutativity of the squares:

$$\begin{array}{ccc} H_n(C_\bullet) & \longrightarrow & H_{n-1}(A_\bullet) \\ \downarrow & & \downarrow \\ H_n(C'_\bullet) & \longrightarrow & H_{n-1}(A'_\bullet) \end{array} .$$

This follows from the fact that if $\alpha(a) = db$ and $\beta(b) = z$, these relations continue to hold when we map $a \in A_{n-1}$, $b \in B_n$ and $z \in C_n$ to their counterparts in A'_{n-1}, B'_n, and C'_n. $\square$

Corollary. *If $0 \to N_2 \to N_1 \to N_0 \to 0$ is a short exact sequence of R-modules and M is any R-module, then there is a long exact sequence*

$$\cdots \to \mathrm{Tor}_n^R(M, N_2) \to \mathrm{Tor}_n^R(M, N_1) \to \mathrm{Tor}_n^R(M, N_0) \to \mathrm{Tor}_{n-1}^R(M, N_2) \to \cdots \to$$

$$\mathrm{Tor}_1^R(M,\,N_2) \to \mathrm{Tor}_1^R(M,\,N_1) \to \mathrm{Tor}_1^R(M,\,N_0) \to M \otimes_R N_2 \to M \otimes_R N_1 \to M \otimes_R N_0 \to 0,$$

where we are identifying $\mathrm{Tor}_0^R(M,\,N)$ *with* $M \otimes_R N$*.*

Moreover, the long exact sequence is functorial in the the short exact sequence

$$0 \to N_2 \to N_1 \to N_0 \to 0.$$

Proof. Let $P_\bullet$ be a projective resolution of M (so that $H_0(P_\bullet) = M$), and let $N_\bullet$ be the short exact sequence formed by the N_i. Then $N_\bullet \otimes_R P_\bullet$ is a double complex that may be thought of as the short exact sequence of complexes

$$0 \to N_2 \otimes_R P_\bullet \to N_1 \otimes_R P_\bullet \to N_0 \otimes_R P_\bullet \to 0.$$

The typical row

$$0 \to N_2 \otimes_R P_n \to N_1 \otimes_R P_n \to N_0 \otimes_R P_n \to\to 0$$

is exact because P_n is projective and, therefore, R-flat. The result is now immediate from the definition of Tor and the snake lemma. $\square$

Note that if P is projective, $\mathrm{Tor}_n^R(P, N) = 0$ for $n \geq 1$. This is obvious because with $P_0 = P$, the complex

$$0 \to P_0 \to 0$$

is a projective resolution of P, and may be used to compute Tor. We shall shortly see that this property, the functorial long exact sequence, and the fact that $\mathrm{Tor}_0^R(M,\,N) \cong M \otimes_R N$ canonically as functors of two variables completely characterizes the functor $\mathrm{Tor}_\bullet^R(\_\,,\ \_)$, up to isomorphism of functors of two variables.

One may ask if there is a comparable long exact sequence for Tor if one starts with a sequence of modules $0 \to M_2 \to M_1 \to M_0 \to 0$. There is such a sequence, and there are several ways to see this. One of them is to prove that there is a canonical isomorphism of functors of two variables $\mathrm{Tor}_n^R(M,\,N) \cong \mathrm{Tor}_n^R(N,\,M)$ for all n, induced by the canonical identification $M \otimes_R N \cong N \otimes_R M$ that lets $u \otimes v$ correspond to $v \otimes u$. But the commutativity of tensor products is not the whole story. The symmetry of Tor is asserting that one can compute $\mathrm{Tor}_n^R(M, N)$ by taking a projective resolution of N, tensoring with M, and then taking homology. It is not obvious how to compare the two. What we shall do is take projective resolutions $P_\bullet$ of M and $Q_\bullet$ of N, and compare the two ways of computing Tor with the homology of $\mathcal{T}_\bullet(P_\bullet \otimes_R Q_\bullet)$. The following fact about double complexes is the key — before stating it, we recall that a left complex is *acyclic* if its homology vanishes in all degrees except degree 0. (The same term is applied to right complexes whose homology vanishes except in degree 0.)

Theorem. *Let $M_{\bullet\bullet}$ be a double complex whose terms all vanish if either component of the bidegree is < 0. Suppose that every row and every column is acyclic, i.e., that the homology of every row is 0 except in degree 0, and the same holds for columns. Let A_i be the augmentation module of the i th row (its 0 th homology module) and B_j be the augmentation module of the j th column (its 0 th homology module). Note that vertical differentials give a map from the i th row to the $i-1$ st row and hence induce maps $A_i \to A_{i-1}$ for all i which makes $A_\bullet$ a complex. Similar, $B_\bullet$ is a complex. Then there are isomorphisms*

$$H_\bullet(A_\bullet) \cong H_\bullet\big(\mathcal{T}_\bullet(M_{\bullet\bullet})\big) \cong H_\bullet(B_\bullet).$$

Proof of the Theorem. Every element of $H_n(\mathcal{T}_\bullet(M_{\bullet\bullet})$ is represented by a cycle of

$$M_{0n} \oplus M_{1,n-1} \oplus \cdots \oplus M_{n-1,0} + \oplus M_{n,0}.$$

Denote this cycle

$$z = u_{0n} \oplus u_{1,n-1} \oplus \cdots \oplus u_{n-1,1} + \oplus u_{n,0}.$$

We work in the signed double complex associated with $M_{\bullet\bullet}$, and assume that horizontal differentials d and the vertical differentials d' anticommute. We shall also write d (respectively, d') for the maps $M_{n,0} \to A_n$ (respectively, $M_{0,n} \to B_n$). A typical term in the sum above has the form u_{ij} where $i + j = n$, and both i and j lie between 0 and n inclusive. The condition that z be a cycle is that for $1 \le i \le n$, $du_{i-1,j+1} = -d'u_{ij}$: this is a condition on the pairs of consecutive terms whose indices sum to n. Given such an element of $\mathcal{T}_n(M_{\bullet\bullet})$, we map it to $H_n(A_\bullet)$ by sending it to $[du_{n0}]$, where $du_{n0} \in A_n$ and the brackets indicate the class of du_{n0} in $H_n(A_\bullet)$. There is a precisely similar map that sends $[z]$ to $[d'(u_{0n})] \in H_n(B_\bullet)$. There are several things that need checking:

(1) du_{n0} is a cycle of $H_n(A_\bullet)$ (the symmetric fact for $[u_{0n}]$ then follows).
(2) $[du_{n0}]$ is independent of the choice of representative of $[z]$ (the symmetric fact for $[d'u_{0n}]$ follows).
(3) The maps $H_n\big(\mathcal{T}_\bullet(M_{\bullet\bullet})\big)$ to $H_n(A_\bullet)$ and to $H_n(B_\bullet)$ obtained in this way are surjective.
(4) These maps are also injective.
(5) These maps are R-linear.

The checks that have some interest are (3) and (4), but we look at them all.

Consider the following diagram, in which the rows are exact, the rightmost squares commute (i.e., d'_* is induced by d'), while other squares, only one of which is shown, anticommute:

$$\begin{tikzcd}
 & M_{n+1,0} \arrow[r, "d"] \arrow[d, "d'"'] & A_{n+1} \arrow[r] \arrow[d, "d'_*"] & 0 \\
M_{n,1} \arrow[r, "d"] \arrow[d, "d'"'] & M_{n,0} \arrow[r, "d"] \arrow[d, "d'"'] & A_n \arrow[r] \arrow[d, "d'_*"] & 0 \\
M_{n-1,1} \arrow[r, "d"] & M_{n-1,0} \arrow[r, "d"] & A_{n-1} \arrow[r] & 0
\end{tikzcd}$$

(1) We have that $d'_*[du_{n0}] = [dd'u_{n,0}] \in A_{n-1}$, and $d'u_{n,0} = -du_{n-1,1}$, and therefore $d'_*[du_{n0}] = [-d^2u_{n-1,1}] = [0] = 0$.

(2) If we change z by adding a boundary in the total complex, $u_{n,0}$ changes by adding a term of the form $du_{n,1} + d'u_{n+1,0}$, where $u_{n,1} \in M_{n,1}$ and $u_{n+1,1} \in M_{n+1,1}$. But $du_{n,1}$ maps to 0 in A_n because $d^2 = 0$, and $d'u_{n+1,0}$ maps to $dd'u_{n+1,0} = d'_*du_{n+1,0}$, the image of $du_{n+1,0} \in A_{n+1}$ in A_n, so that $[du_{n,0}]$ does not change.

(3) Suppose that $\zeta \in A_n$ is a cycle. We can write ζ in the form $du_{n,0}$ for some $u_{n,0} \in M_{n,0}$. We want to show that we can construct elements $u_{n-j,j}$, $1 \le j \le n$, such that

$$u_{0n} \oplus u_{1,n-1} \oplus \cdots \oplus u_{n-1,1} + \oplus u_{n,0}$$

is a cycle in $\mathcal{T}_n(M_{\bullet\bullet})$, i.e., such that we have

$$(*_j) \quad du_{n-(j+1),j+1} = -d'u_{n-j,j}$$

$0 \le j \le n-1$, and we proceed to make the construction by induction on j. Because $du_{n,0} = \zeta$ is a cycle, $d'_*du_{n,0} = 0$, which implies $dd'u_{n,0} = 0$. Since $-d'u_{n,0}$ is in the kernel of d, it is in the image of d, and so we can choose $u_{n-1,1} \in M_{n-1,1}$ such that $du_{n-1,1} = -d'u_{n,0}$. This is $(*_0)$. Now suppose that the $u_{n-h,h}$ have been constructed such that $(*_{h-1})$ holds, $1 \le h \le j$, where $j \ge 1$. In particular, we have $(*_{j-1})$, i.e.,

$$du_{n-j,j} = -d'u_{n-j+1,j-1}.$$

We want to choose $u_{n-j+1,j+1}$ such that

$$du_{n-(j+1),j+1} = -d'u_{n-j,j}$$

so that it suffices to see that $-d'u_{n-j,j}$ is in the image of d, and, therefore, it suffices to see that it is in the kernel of d. but

$$-dd'u_{n-j,j} = d'du_{n-j,j} = d'(-d'u_{n-j+1,j-1}) = 0,$$

as required, since $(d')^2 = 0$. This shows that one can construct a cycle that maps to ζ. If we let $w_{n-j-1,j} = d'u_{n-j,j}$, we have this picture:

$$
\begin{array}{ccccccccccc}
 & & & & & & & & u_{n,0} & \mapsto & z \\
 & & & & & & & & \mapsdown & & \\
 & & & & & & u_{n-1,1} & \mapsto & \pm w_{n-1,0} & & \\
 & & & & & & \mapsdown & & & & \\
 & & & & & & \pm w_{n-2,1} & & & & \\
 & & & \iddots & & \iddots & \iddots & & & & \\
 & & u_{n-j,j} & \mapsto & \pm w_{n-j,j-1} & & & & & & \\
 & & \mapsdown & & & & & & & & \\
u_{n-j-1,j+1} & \mapsto & \pm w_{n-j-1,j} & & & & & & & & \\
\mapsdown & & & & & & & & & & \\
\pm w_{n-j-2,j+1} & & & & & & & & & &
\end{array}
$$

(4) Now suppose that we have a cycle in $\mathcal{T}_n(M_{\bullet\bullet})$, call it

$$z = u_{0n} \oplus u_{1,n-1} \oplus \cdots \oplus u_{n-1,1} + \oplus u_{n,0}$$

that maps to 0 in $H_n(A_\bullet)$, which means that $du_{n,0} \in A_n$ is the image of some $a_{n+1} = du_{n+1,0} \in A_{n+1}$ under the map induced by d'. This implies that $d(u_{n,0} - d'u_{n+1,0}) = du_{n,0} - d'_* du_{n+1,0} = 0$ in A_n, and therefore has the form $du_{n,1}$ for some $u_{n,1} \in M_{n,1}$. We now use recursion on j to construct

$$u_{n-1,2} \in M_{n-1,2},\ \ldots,\ u_{n-j,j+1} \in M_{n-j,j+1},\ \ldots,\ u_{0,n+1} \in M_{0,n+1}$$

such that for all j, $0 \le j \le n$,

$$(*_j) \quad du_{n-j,j+1} + d'u_{n-j+1,j} = u_{n-j,j}.$$

This will show that z is the image of

$$u_{0,n+1} \oplus u_{1,n} \oplus \cdots \oplus u_{n,1} \oplus u_{n+1,0},$$

as required. We have already done the case where $j = 0$. Suppose for a fixed j with $1 \le j \le n$ we have constructed these elements $u_{n+1-h,\,h}$, $0 \le h \le j$, such that $(*_h)$ holds for $0 \le h \le j-1$. In particular, for $h = j-1$, we have

$$(*_{j-1}) \quad du_{n-j+1,j} + d'u_{n-(j-1)+1,j-1} = u_{n-j+1,j-1},$$

and applying d' to both sides we get:

$$(**) \quad d'du_{n+1-j,j} = d'u_{n+1-j,j-1}$$

We want to construct $u_{n-j,j+1}$ such that $(*_j)$ holds, i.e., such that

$$du_{n-j,j+1} = u_{n-j,j} - d'u_{n+1-j,j}.$$

To show that the element on the right is in the image of d, it suffices to prove that it is in the kernel of d, i.e., that

$$du_{n-j,j} = dd'u_{n+1-j,j}.$$

But $du_{n-j,j} = -d'u_{n-j+1,j-1}$ because z is a cycle and by $(**)$,

$$-d'u_{n+1-j,j-1} = -d'du_{n+1-j,j} = dd'u_{n+1-j,j},$$

as required.

(5) R-linearity is immediate from the definitions of the maps, once we know that they are well-defined, since, at the cycle level, the map $H_n\big(\mathcal{T}_\bullet(M_{\bullet\bullet})\big) \to H_n(A_\bullet)$ is induced by restricting the product projection $\prod_{i+j=n} M_{ij} \to M_{n0}$ (identifying $\bigoplus_{i+j=n} M_{ij} \cong \prod_{i+j=n} M_{ij}$). $\square$

We immediately obtain the isomorphism $\mathrm{Tor}_n^R(M, N) \cong \mathrm{Tor}_n^R(N, M)$ for all n. Let $P_\bullet$ and $Q_\bullet$ be projective resolutions of M and N, respectively. Then $\mathrm{Tor}_n^R(M, N) \cong H_n(P_\bullet \otimes_R N) \cong H_n\big(\mathcal{T}_\bullet(P_\bullet \otimes_R Q_\bullet)\big) \cong H_n(M \otimes_R Q_\bullet) \cong H_n(Q_\bullet \otimes_R M) \cong \mathrm{Tor}_n^R(N, M)$. The first and last isomorphisms follow from the definition of Tor, coupled with the fact that any projective resolution may be used to compute it, the second and third isomorphisms follow from the Theorem just proved, and the next to last isomorphism is a consequence of the commutativity of tensor product.

This means that given a short exact sequence of modules $0 \to M_2 \xrightarrow{a} M_1 \xrightarrow{b} M_0 \to 0$ there is also a long exact sequence for Tor:

$$\cdots \to \mathrm{Tor}_n^R(M_2, N) \to \mathrm{Tor}_n^R(M_1, N) \to \mathrm{Tor}_n^R(M_0, N) \to \mathrm{Tor}_{n-1}^R(M_2, N) \to \cdots.$$

This sequence can be derived directly without proving the commutativity of Tor, by constructing an exact sequence of projective resolutions of the modules M_j instead. The idea is to fix resolutions of M_2 and M_0, and use them to build a resolution of M_1. Suppose that we are given projective resolutions $P_\bullet^{(j)}$ of M_j, for $j = 2, 0$, and call the differentials $d^{(j)}$, $j = 0, 2$. From these we can construct a projective resolution $P_\bullet^{(1)}$ of M_1 such that for all n, $P_n^{(1)} = P_n^{(2)} \oplus P_n^{(0)}$. To begin, the map $d^{(0)} : P_0^{(0)} \to M_0$ lifts to a map $f_0 : P^{(0)} \to M_1$ by the universal mapping property of projective modules, because $b : M_1 \twoheadrightarrow M_0$ is onto.

One gets a surjection $d^{(1)} : P^{(2)} \oplus P^{(0)} \twoheadrightarrow M_1$ using $d^{(1)} = a \circ d^{(2)} \oplus f_0$. If one lets Z_2, Z_1, and Z_0 be the kernels of the $d^{(j)}$ one has a commutative diagram:

$$\begin{tikzcd}
 & P_1^{(2)} \arrow[d] & & P_1^{(0)} \arrow[d] & \\
0 \arrow[r] & Z_2 \arrow[r] \arrow[d] & Z_1 \arrow[r] \arrow[d] & Z_0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P_0^{(2)} \arrow[r] \arrow[d, "d^{(2)}"'] & P_0^{(2)} \oplus P_0^{(0)} \arrow[r] \arrow[d] & P_0^{(0)} \arrow[r] \arrow[d, "d^{(0)}"] & 0 \\
0 \arrow[r] & M_2 \arrow[r, "a"'] \arrow[d] & M_1 \arrow[r, "b"'] \arrow[d] & M_0 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 & .
\end{tikzcd}$$

where the sequence of kernels $0 \to Z_2 \to Z_1 \to Z_0 \to 0$ is easily checked to be exact, and the problem of constructing the degree 1 part of the resolution of M_1 is now precisely the same problem that we had in constructing the degree 0 part.

Once one has the map $P_1^{(1)} = P_1^{(2)} \oplus P_1^{(0)} \twoheadrightarrow Z_1$, the map

$$P_1^{(1)} = P_1^{(2)} \oplus P_1^{(0)} \to P_0^{(2)} \oplus P_0^{(0)} = P_0^{(1)}$$

is constructed as the composition of the map $P_1^{(2)} \oplus P_1^{(0)} \twoheadrightarrow Z_1$ with the inclusion of Z_1 in $P_0^{(2)} \oplus P_0^{(0)}$. By a straightforward induction, one can continue in this way to build an entire projective resolution $P_\bullet^{(1)}$ of M_1, and a short exact sequence of complexes

$$0 \to P_\bullet^{(2)} \to P_\bullet^{(1)} \to P_\bullet^{(0)} \to 0$$

such that for all n,

$$P_n^{(1)} = P_n^{(2)} \oplus P_n^{(0)},$$

and the induced sequence of maps on the augmentations M_j is the short exact sequence $0 \to M_2 \xrightarrow{a} M_1 \xrightarrow{b} M_0 \to 0$ that we started with.

We next note that if $r \in R$ and M, N are R-modules, then the map

$$\operatorname{Tor}_n^R(M, N) \to \operatorname{Tor}_n^R(M, N)$$

induced by multiplication by r on N is given by multiplication by r on $\operatorname{Tor}_n^R(M, N)$. This may be seen as follows. Choose a projective resolution $P_\bullet$ of M. When we tensor with

$N \xrightarrow{r} N$, we get the map of complexes $P_\bullet \otimes_R N \xrightarrow{r} P_\bullet \otimes_R N$ induced by multiplication by r, and this induces the map of homology. The same fact holds when we use $M \xrightarrow{r} M$ to induce a map

$$\mathrm{Tor}_n^R(M,\, N) \to \mathrm{Tor}_n^R(M,\, N),$$

by the symmetry of Tor. (Alternatively, use multiplication by r on every P_n to left $M \xrightarrow{r} M$ to a map $P_\bullet \xrightarrow{r} P_\bullet$ of the projective resolution of M to itself. Then apply $\_ \otimes_R N$ and take homology.)

If $r \in \mathrm{Ann}_R N$, then multiplication $N \xrightarrow{r} N$, is the zero map, and hence induces the 0 map

$$\mathrm{Tor}_n^R(M,\, N) \to \mathrm{Tor}_n^R(M,\, N),$$

which is also the map given by multiplication by r. In consequence, we have that $\mathrm{Ann}_R N$ kills every $\mathrm{Tor}_n^R(M,\, N)$. The same holds for $\mathrm{Ann}_R M$, and so $\mathrm{Ann}_R M + \mathrm{Ann}_R N$ kills every $\mathrm{Tor}_n^R(M,\, N)$.

The following fact, while very simple, is of great utility:

Proposition. *If $x \in R$ is not a zerodivisor and M is any R-module, then $\mathrm{Tor}_n^R(M,\, R/xR)$ (which is also $\mathrm{Tor}_n^R(R/xR,\, M)$) is M/xM if $n = 0$, is $Ann_M x$ if $n = 1$, and is 0 if $n \neq 0,\, 1$.*

Proof. We may use the projective resolution $0 \to R \xrightarrow{x} R \to 0$, whose augmentation is R/xR, to compute Tor. Here, the left hand copy of R is in degree 1 and the right hand copy in degree 0. When we apply $M \otimes_R \_$, we find that the values of Tor are given by the homology of the complex $0 \to M \xrightarrow{x} M \to 0$. $\square$