Faithful Flatness

We shall say that an R-module F is *faithfully flat* if it is flat if it is flat and for every nonzero R-module M, $F \otimes_R M \neq 0$. An R-algebra S is *faithfully flat* if it is faithfully flat when considered as an R-module. We shall see below that the completion of a local ring R is a faithfully flat R-algebra. Typically, $W^{-1}R$ is flat but not faithfully flat: if W contains an element that is not already a unit, say f, then $W^{-1}R \otimes_R (R/fR) = 0$. A nonzero free module over R is obviously faithfully flat.

Proposition. *Let F be an R-module. The following conditions are equivalent:*
(1) *F is flat and for every nonzero R-module M, $F \otimes M_R \neq 0$ (i.e., M is faithfully flat).*
(2) *F is flat and for every proper ideal I of R, $IF \neq F$.*
(3) *F is flat and for every maximal ideal m of R, $mF \neq F$.*
(4) *F is flat and for every R-linear map $h : M \to N$, h is nonzero if and only if $\mathrm{id}_F \otimes h : F \otimes_R M \to F \otimes_R N$ is nonzero.*
(5) *For every sequence of modules $A \to B \to C$, the sequence is exact at B if and only if the sequence $F \otimes_R A \to F \otimes_R B \to F \otimes_R C$ is exact at $F \otimes_R B$.*

Proof. The conclusion in (2) is equivalent to $F/IF = F \otimes_R (R/I) \neq 0$. Therefore, (1) $\Rightarrow$ (2) $\Rightarrow$ (3). Now assume (3) and let M be any nonzero module. Then M has a nonzero element u. Let $I = Ann_R u$, so that $Ru \cong R/I$. Let m be a maximal ideal containg I. Since $IF \subseteq mF \neq F$, we have that $F \otimes_R R/I \neq 0$. Since $R/I \cong Ru \hookrightarrow M$ and F is flat, we have that $F/IF \hookrightarrow F \otimes_R M$, so that $F \otimes_R M \neq 0$. Thus, (3) $\Rightarrow$ (1). This shows that (1), (2), and (3) are equivalent.

In (4), the "if" part is obvious. If we apply (4) to the map $0 \to M$, we see that (4) $\Rightarrow$ (1). We need to show if (1) holds, the "only if" part of (4) holds. Suppose that $M \to N$ factors $M \twoheadrightarrow Q \hookrightarrow N$, where Q is the image of N. The map is nonzero if and only if $Q \neq 0$. Then $F \otimes_R M \to F \otimes_R N$ factors $F \otimes_R M \twoheadrightarrow F \otimes_R Q \hookrightarrow F \otimes_R N$, where the map on the left is surjective by the right exactness of $\otimes$, and the map on the right is injective because F is flat. By (1), we have that $F \otimes_R Q \neq 0$.

The fact that "only if" part of (5) holds implies that $F \otimes_R \_$ preserves short exact sequences, which is equivalent to the flatness of F. Therefore, in the rest of the argument we may assume that F is flat.

To see that (5) is equivalent to the other conditions, let f, g respectively denote $A \to B$ and $B \to C$. Let D be the image of A in B. Let E be the kernel of the map $B \to C$, and let G be the image of $B \in C$. This, we have $A \twoheadrightarrow D$, $D \hookrightarrow B$, $E \hookrightarrow B$, $0 \to E \to B \to G \to 0$ is exact, and $G \hookrightarrow C$. Because F is flat, all these conditions are preserved when apply $F \otimes_R \_$. This means that we may identify the image of $F \otimes f$ with $F \otimes_R D$, and the kernel of $F \otimes g$ with $F \otimes_R E$. The original sequence is exact at B if and only if $D = E$. Obvious, this implies that we have exactness when we apply $F \otimes_R \_$: this only uses that F is flat. It remains to show that if the images of $F \otimes D$ and $F \otimes E$ are equal in $F \otimes B$, then $D = E$. But if the images are equal, they will both be equal to the image of $F \otimes (D+E)$. Then $F \otimes \big((D+E)/D\big) \cong \big(F \otimes (D+E)\big)/\mathrm{Im}\,(F \otimes D) = 0$, which shows that $(D+E)/D = 0$ by (1), and hence that $D+E = D$. But $D+E = E$ follows in exactly the same way. $\square$

In the situation of the Corollary below, m is the only maximal ideal of R, and $mS \neq S$ if and only if m maps into n.

Corollary. *A flat homomorphism $h : (R,m) \to (S,n)$ of quasilocal rings is faithfully flat if and only if it is local, i.e., if and only if m maps in to n.* $\square$

Proposition. *If M is flat (respectively, faithfully flat) over R and T is any R-algebra, $T \otimes_R M$ is flat (respectively, faithfully flat) over T.*

Proof. If $f : A \to B$ is a map of T-modules, we may use the associativity of $\otimes$ to identify $A \otimes_T (T \otimes_R M) \to B \otimes_T (T \otimes_R M)$ with the map $A \otimes_R M \to B \otimes_R M$. Thus, if f is injective, the flatness of M over R implies the new map is injective, while if A is nonzero, so is $A \otimes_T (T \otimes_R M) \cong A \otimes_R M$. $\square$

Proposition. *If S is a faithfully flat R algebra and I is an ideal of R, then the contraction of IS to R is I. Moreover, $R \to S$ is injective.*

Proof. For any ideal $\mathfrak{A}$ of R, we have an injection $\mathfrak{A} \hookrightarrow R$, which yields an injection $\mathfrak{A} \otimes S \hookrightarrow S$ when we apply $\_ \otimes_R S$. The image of the injection is $\mathfrak{A}S$, so that $\mathfrak{A} \otimes_R S \cong \mathfrak{A}S$. If $\mathfrak{A}$ is the kernel of $R \to S$, we then have $\mathfrak{A} \otimes S \cong \mathfrak{A}S = 0$. Since S is faithfully flat, this implies $\mathfrak{A} = 0$. This proves the second statement. But then for every I, the preceding result shows that $R/I \to S/IS$ is faithfully flat (take $M = S$, $T = R/I$), and so injective. The kernel is J/I, where J is the contraction of IS to R, and so $J = I$. $\square$

Corollary. *If $R \to S$ is faithfully flat, then* $\mathrm{Spec}\,(S) \to \mathrm{Spec}\,(R)$ *is surjective.*

Proof. For every prime P of R, the contraction of PS to R is P, which means that PS is disjoint from the image W of $(R-P)$ in S, which is a multiplicative system. Hence, there is a prime Q of S that contains PR and is disjoint from W, and Q must contract to P. $\square$

Proposition. *Let $(R,m) \to (S,n)$ be a flat local homomorphism of local rings. Then* $\dim\,(S) = \dim\,(R) + \dim\,(S/mS)$. *($S/mS$ is called the* closed fiber *of $R \to S$.)*

Proof. We use induction on $\dim\,(R)$. If $J = \mathrm{Rad}\,(0)$ in R, $R/J \to S/JS$ is again flat and local, and, since both J and JS consist of nilpotents, the dimensions do not change (note that the closed fiber also has not changed.) Therefore, we may assume that R is reduced. If $\dim\,(R) = 0$, then R is a field, $m = 0$, and $S/mS \cong S$, so the result is clear. Otherwise, m is not contained in the union of the minimal primes of R: choose $x \in m$ not in any minimal prime. Since R is reduced, every associated prime of (0) is minimal. Hence, x is not a zerodivisor in R. Since $R \xrightarrow{x\cdot} R$ is injective, when we apply $S \otimes_R \_$ we obtain an injection $S \xrightarrow{x\cdot} S$. Thus, $\dim\,(R/xR) = \dim\,(R) - 1$, and $\dim\,(S/xS) = \dim\,(S) - 1$. But $R/xR \to S/xS$ is still flat local with the same closed fiber. By the induction hypothesis, $\dim\,(S) - 1 = \dim\,(R) - 1 + \dim\,(S/mS)$ and the result follows. $\square$