Math 615, Winter 2014 **Problem Set #1**

Due: January 31

1. Let R be any domain that is $\mathbb{N}$-graded. Let I_n be the ideal of R generated by forms of degree $\geq n$. Prove that I is integrally closed.

2. Let I be an integrally closed ideal of R. Let $R[x]$ be a polynomial ring in one variable over R. Prove that $IR[x]$ is integrally closed, and that $IR[x] + xR[x]$ is integrally closed.

3. Let $R = K[x_1, \ldots, x_n]$ be a polynomial ring over a field K. If $\alpha = (a_1, \ldots, a_n) \in \mathbb{N}^n$, let $x^\alpha = x_1^{a_1} \cdots x_n^{a_n}$, and let $L(x^\alpha) = \alpha$. If I is an ideal generated by monomials, let $L(I) = \{\alpha \in \mathbb{N}^n : x^\alpha \in I\}$. Note that there is a bijection between monomial ideals in R and subsets S of $\mathbb{N}^n$ such that if $\alpha \in S$ and $\beta \in \mathbb{N}^n$ then $\alpha + \beta \in S$. Show that I is integrally closed if and only if $L(I)$ contains every point of $\mathbb{N}^n$ in its convex hull over the rational numbers $\mathbb{Q}$.

4. Show that if the subring $R \subseteq S$ is integrally closed in S and W is a multiplicative system in R, then $W^{-1}R$ is closed in $W^{-1}S$. In particular a localization of a normal ring is normal.

5. (Integral closure of ideals commutes with localization.) Let $I \subseteq R$ be an ideal, and W a multiplicative system of R. Show that the integral closure of $IW^{-1}R$ in $W^{-1}R$ is the expansion of $\overline{I}$ to $W^{-1}R$.

6. Show that if $I_1, \ldots, I_n$ are ideals of R, then $\overline{I_1}\,\overline{I_2} \cdots \overline{I_n} \subseteq \overline{I_1 I_2 \cdots I_n}$. Hence, for an ideal $I \subseteq R$, $\overline{I}^n \subseteq \overline{I^n}$.

Extra Credit 1. Let R be a commutative ring such that R_m is a domain for every maximal ideal m and such that $\mathrm{Spec}\,(R)$ is connected. Must R be a domain? Prove your answer. (R is not assumed Noetherian: the statement is true if R is Noetherian.)

Extra Credit 2. Let I be an integrally closed monomial ideal in a polynomial ring $K[x_1, \ldots, x_n]$. Let $m = (x_1, \ldots, x_n)R$. Is mI necessarily integrally closed? Prove your answer.