Math 615, Winter 2014 **Problem Set #3**

Due: Friday, March 14 (π day)

1. $S = K[x_1, \ldots, x_n]$ is polynomial over the field K. $R = K[x_i^3, x_ix_j : 1 \leq i, j \leq n] \subseteq S$. Determine, explicitly, the conductor of R in S.

2. $S = K[x_1, \ldots, x_n]$ is polynomial over the field K, and $R = K[F : \deg(F) = d] = K[S_d] \subseteq S$, where the characteristic of K is 0 or is $> d$. Find $\mathcal{J}_{S/R}$ explicitly.

3. Let (R, m) be the localization of the polynomial ring $\mathbb{R}[x, y]$ over the real numbers $\mathbb{R}$ at the maximal ideal (x, y). Let (V, n) be the DVR obtained by localizing the polynomial ring $\mathbb{R}[t, z]$ at the prime ideal generated by $u = z^2 + 1$. Since $z^2 + 1$ and$(z^2 + 1)zt$ are algebraically independent over $\mathbb{R}$, we have an injection $K[x, y] \hookrightarrow V$ that sends $x \mapsto u$ and $y \mapsto uzt$. Since (x, y) maps into n, this yields an inclusion $R \hookrightarrow V$. Describe the quadratic sequence (T_i, m_i, K_i), where $T_0 = R$ and T_h is a DVR, of R along V. For each T_i, what is its Krull dimension, and what is K_i as a subfield of $\mathbb{C}(t)$?

4. Let (R, m, K) be a regular local ring of dimension $d \geq 2$. Let ord denote the m-adic order of an element of R: if $r \in m^h - m^{h+1}$, then $\mathrm{ord}\,(r) = h$. You may assume that ord is a valuation, and let V be the corresponding DVR of $\mathrm{frac}\,(R)$. Describe the quadratic sequence of R along V. Is V necessarily essentially of finite type over R?

5. Let $V = K[[x]]$, where K is a field, and let f be an element of xV such that x and f are algebraically independent over K. (E.g., one might take K to be $\mathbb{Q}$, $\mathbb{R}$, or $\mathbb{C}$, and f to be $\sin(x)$ or $e^x - 1$. Such elements exist no matter what K is.) Note that we have a $K[x]$-algebra injection of the polynomial ring $K[x, y] \hookrightarrow V$ such that $y \mapsto f$. Then (x, y) maps into xV, so that if (R, m) is the localization of $K[x, y]$ at (x, y), we have $R \hookrightarrow V$. Consider the quadratic sequence (T_i, m_i, K_i) of R along V. Show that $K_i = K$ for all i. What is the sequence of Krull dimensions of the T_i? Is the sequence of T_i finite? Is the DVR $W = \bigcup_{i=0}^{\infty} T_i$ essentially of finite type over R?

6. Let R be a normal Noetherian domain that is finitely generated over a field K, and let $I_1, \ldots, I_n$ be ideals of R. Show that there is a $c \in \mathbb{N}$ such that for all $h_1, \ldots, h_n \in \mathbb{N}$, $\overline{\prod_{j=1}^{n} I_j^{h_j}} = \prod_{\{j : h_j \geq c\}} I_j^{h_j - c} \overline{\prod_j I_j^{\min\{h_j, c\}}}$.

Extra Credit 5. Let $S = K[x_1, \ldots, x_n]/(F_1, \ldots, F_m)$ be a domain, where K is a field of characteristic 0. Let J be the ideal generated by the $(n-d) \times (n-d)$ minors of the matrix $\mathcal{M} = \big(\partial F_j / \partial x_i\big)$. Let S' be the integral closure of S. Show that $JS' \subseteq S$. (Make a linear change of coordinates to generalize the position of the x_i. The new Jacobian matrix may be obtained from the original by multiplying by an invertible matrix of scalars: the ideals of minors don't change. After this change, S will be module-finite over the polynomial subring generated by any d of the x_i, say $x_{i_1}, \ldots, x_{i_d}$. Now the F_j may be thought of as giving a presentation of S over $R[x_{i_1}, \ldots, x_{i_d}]$ using the remaining $n - d$ variables as generators. Apply the Lipman-Sathaye theorem.)

Extra Credit 6. Let $A \subseteq R$, $B \subseteq S$ be finitely generated domains over a field K such that $R \otimes_K S$ is normal. Show that R and S are normal. Let R be module-finiite over A with conductor $\mathfrak{C}$ and S be module-finite over B with conductor $\mathfrak{D}$. Is the conductor of $A \otimes_K B$ in $R \otimes_K S$ determined by $\mathfrak{C}$ and $\mathfrak{D}$? Prove your answer.