Math 615, Winter 2014 **Problem Set #4**
Due: Friday, March 28

1. Let $R = K[x_1, \ldots, x_n]$ be a polynomial ring over a field K and let S denote the K-algebra $K[x_1, x_2/x_1, \ldots, x_n/x_1]$. Determine $\mathcal{J}_{S/R}$ and $W_{S/R}$.

2. Let $n \geq 1$ be an integer, let $R = K[x, y]$ and let $S = K[X^4, X^3Y, XY^3, Y^4]$, and think of R as a subring of S using the K-algebra injection such that $x \mapsto X^4$ and $y \mapsto Y^4$. Assume that $\mathrm{char}(K) \neq 2$. Let S' denote the integral closure of S. Determine $W_{S/R}$ and $W_{S'/R}$ as submodules of $\mathcal{L} = \mathrm{frac}\,(S)$.

3. Let S_1 and S_2 be finitely generated, torsion-free, and generically étale extensions of polynomial rings $R_1 = K[x_1, \ldots, x_m]$ and $R_2 = K[y_1, \ldots, y_n]$, respectively, let $S = S_1 \otimes_K S_2$, and let $R = R_1 \otimes_K R_2$. Show that S is finitely generated, torsion-free, and generically étale over R.

4. With the same hypothesis as in **3.**, determine whether $W_{S/R}$ can be identified with $W_{S_1/R_1} \otimes_K W_{S_2/R_2}$.

5. Let R be a regular ring of positive characteristic p, and let I, J be ideals of R. Let $q = p^n$ for some positive integer n. Prove that $I^{[q]} :_R J^{[q]} = (I :_R J)^{[q]}$. (Suggestion: first show that if S is flat over R (which need not be regular) and J is finitely generated, $IS :_S JS = (I :_R J)S$.)

6. Let K be a field of positive characteristic p and let $R = K[x^2, xy, y^2] \subseteq K[x, y]$, where x, y are indeterminates. Determine the Hilbert-Kunz multiplicities of the ideals I_1, I_2, J_1, J_2 and J_3 of R, where $I_1 = (x^4, y^4)$, $I_2 = I_1 + (x^3y^3)$, $J_1 = I_2 + (x^3y)$, $J_2 = I_2 + (x^2y^2)$, and $J_3 = I_2 + (xy^3)$.

7. Let R be a Noetherian ring of prime characteristic $p > 0$. Let I be an ideal such that $I = I^*$, let $f \in R$ and $J = I :_R fR$. Must $J = J^*$? Prove your answer.

Extra Credit 7. Let R be a regular Noetherian domain of characteristic $p > 0$ and let S be a module-finite domain extension of R such that the extension of fraction fields is separable. If D is domain of characteristic p, let $D^{1/q} = \{d^{1/q} : d \in D\}$. Show that $\mathcal{J}_{S/R}S^{1/q} \subseteq R^{1/q}[S]$ for all q.

Extra Credit 8. Let M and N be finitely generated modules over a Noetherian ring R. Give as an informative characterization as you can of the associated primes of $\mathrm{Hom}_R(M, N)$.