Math 615, Winter 2014 **Problem Set #5**
Due: Friday, April 18

Throughout, R and S are Noetherian domains of positive characteristic p.

1. Let $S = K[W, X, Y, Z]$, a polynomial ring over a field K of characteristic $p > 2$, and let $R = S/(W^4 + X^4 + Y^4 + Z^4) = K[w, x, y, z]$. Let $I = (w, x, y)R$. Show that for all $p > 2$, $z^3 \in I^* - I$. For which p is it true that $(z^3)^p \in I^{[p]}$?

2. Let $x_1, \ldots, x_n$ be indeterminates over a field K of characteristic p. Let d be a positive integer and let $R^{(d)}$ denote the subring of $K[x_1, \ldots, x_n]$ spanned over K by all forms whose degree is a nonnegative integer multiple of d. Let $\mathcal{M}_d$ denote the homogeneous maximal ideal of $R^{(d)}$. Let t be a positive integer. What is the Hilbert-Kunz multiplicity of $\mathcal{M}_d^t$ in $R^{(d)}$? Your answer should be a function of n, d, and t.

3. Let $R \subseteq S$ and suppose that there is an R-linear map $\theta : S \to R$ such that $\theta(1) \neq 0$. Prove that for every ideal I of R, $IS \cap R \subseteq I^*$.

4. Let $I \subseteq R$ be any ideal and let $n > 0$ be an integer. Show that $(I^*)^{[p^n]} \subseteq (I^{[p^n]})^*$.

5. Let $\tau(R)$ be defined as $\displaystyle\bigcap_{I \subseteq R} I :_R I^*$. Show from this definition that if $c \in \tau(R)$ and $f \in I^*$ for some I, then $cf^{p^n} \subseteq I^{[p^n]}$ for all $n \in \mathbb{N}$. (This shows that two possibly different notions of test ideal agree.) Thus, any nonzero element c_0 of $\tau(R)$ can be used in testing tight closure: if there exists $c \neq 0$ in R such that $cf^{p^n} \in I^{[p^n}$ for all $n \gg 0$, then $c_0 f^{p^n} \in I^{[p^n]}$ for all $n \geq 0$.

6. Let $R \subseteq S$ and suppose that for all $I \subseteq R$, $IS \cap R = I$. Prove that $\tau(S) \cap R \subseteq \tau(S)$.

Extra Credit 9. Define the *Frobenius closure* I^{F} of $I \subseteq R$ to be the set of all elements $r \in R$ such that for some positive integer n, $r^{p^n} \in I^{[p^n]}$. Let $c \neq 0$ be an element of R and suppose that for every maximal ideal m of R, $c \in \mathrm{Rad}\,(\tau(R_m))$. Show that for every ideal I of R, $cI^* \subseteq I^{\mathrm{F}}$.

Extra Credit 10. Let R be a finitely generated $\mathbb{N}$-graded Cohen-Macaulay domain of Krull dimension d over a field K of characteristic $p > 0$ with $R_0 = K$ such that every homogeneous system of parameters (i.e., d homogeneous elements of $\mathcal{M} = \bigoplus_{k=1}^{\infty} R_j$ that generate an ideal primary to $\mathcal{M}$) generates a tightly closed ideal. Let $F_1, \ldots, F_d$ be a homogeneous system of parameters of R. Let G be a nonzero homogeneous element of R that is not $I = (F_1, \ldots, F_d)$. Prove that $\deg(G) \leq \sum_{h=1}^{d} \deg(F_h)$.