TIGHT CLOSURE IN EQUAL CHARACTERISTIC, BIG COHEN-MACAULAY ALGEBGRAS, AND SOLID CLOSURE

by Melvin Hochster

Abstract. We first discuss joint work of Craig Huneke and the author, giving an overview of the status of tight closure theory both in characteristic p and in equal characteristic 0, including recently discovered interconnections with the existence of big Cohen-Macaulay algebras, especially their existence in a weakly functorial sense. For example, either tight closure or the functorial existence of big Cohen-Macaulay algebras can be used to prove that direct summands of regular rings are Cohen-Macaulay in equal characteristic. Later we present and explore a new notion, solid closure, defined *a priori* in all characteristics, even mixed characteristic, but agreeing with tight closure in positive characteristic.

PROLOGUE

Let A be a complete local domain. We can define an apparently frivolous closure operation on ideals of A as follows. If $a_1, \dots, a_n \in A$ let $y_1, \dots, y_n$ be new formal variables over A and define $a \in A$ to be in the *closure* of the ideal $(a_1, \dots, a_n)A$ if the power series

$$(\#) \qquad a y_1 \cdots y_n - \sum_{i=1}^{n} a_i y_1 \cdots y_{i-1} y_{i+1} \cdots y_n$$

has a nonzero multiple in $A[[y_1, \dots, y_n]]$ in which none of the monomials occurring involves all of the y's. This condition is independent of the choice of generators of the ideal. It is true that if $a = \sum_{i=1}^{n} r_i a_i$ is in the ideal, then it satisfies this condition: the difference (#) can be rewritten as

$$\sum_{i=1}^{n} (r_i y_i - 1) a_i y_1 \cdots y_{i-1} y_{i+1} \cdots y_n$$

and multiplication by $\prod_i (r_i y_i - 1)^{-1}$ will have the desired effect. It is an incredibly subtle problem to determine whether an element of A is in the closure of an ideal in this sense. But is it important? Yes!!! Let V be the p-adic integers, where p is a positive prime integer, and let $A = V[x_2, x_3]$. Let $x_1 = p$. Then it is not even known whether $x_1^2 x_2^2 x_3^2$ is in the closure of the ideal (x_1^3, x_2^3, x_3^3). On the other hand, if $A = K[[x_1, x_2, x_3]]$ with

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K a field, it is known that $x_1^2x_2^2x_3^2$ is not in the closure of (x_1^3, x_2^3, x_3^3) if K has positive characteristic p, but, by a very recent result of P. Roberts [Ro6], it is in the closure of the ideal generated by the cubes when K has characteristic zero.

In the course of this paper I will explain how one is led to ask this apparently innocuous, somewhat strange, but, in fact, quite difficult and important question. There is a concluding discussion in (6.8).

Problem for thought. Let f, g, h be three elements in a regular ring of dimension two. Is it the case that $(fgh)^2$ must be in (f^3, g^3, h^3)? What if the ring is $\mathbb{C}[[x, y]]$? $\mathbb{C}[x, y]$? Must fgh be in (f^2, g^2, h^2) in these rings?

1. INTRODUCTION

Except where otherwise indicated, all of the results on tight closure in equal characteristic and the existence of big Cohen-Macaulay algebras discussed here are the fruit of a more than six year collaboration of the author with Craig Huneke. The reader is referred to [HH1-12], [AHH], [Hu1,3] for the detailed account, as well as to [Ab1-3], [FeW], [Fe], [Gla], [Hu2], [Sm1-3], [Sw1-2], [Vel], [W2] and [Wil] for additional information and applications. The results on solid algebras and solid closure discussed in §6 were developed by the author in [Ho8] as part of a rather speculative assault on the problem of defining an analogue of tight closure in mixed characteristic. The notion of solid closure does agree with the notion of tight closure in characteristic p in good cases, and does make sense quite generally. It has many of the right properties to be a good analogue of tight closure. By a recent result of Paul Roberts [Ro6] it does not have the right properties in equal characteristic zero. The question of whether it has the right properties in mixed characteristic remains open, although Roberts' result is discouraging. In particular, if it could be proved that every ideal of a regular local ring of mixed characteristic is solidly closed, the direct summand conjecture, in a greatly strengthened form, would follow. This appears to be a difficult question. What we know at the moment is that ideals of regular rings are solidly closed in positive characteristic, not necessarily so in equal characteristic zero, despite which, the issue remains open in mixed characteristic.

We begin by summarizing the properties of tight closure in the equal characteristic case, without even giving the definition. Later, we shall backtrack and give a definition. But before we do that we explain briefly how a number of applications are obtained from the properties that we state.

This approach will enable us to see quickly the merits and defects of the theory of solid closure when it is developed.

Before discussing the theory of solid closure we shall spend some time explaining how the investigation of tight closure led to the discovery of the existence of big Cohen-Macaulay algebras, and how the two theories intertwine.

2. BASIC PROPERTIES OF TIGHT CLOSURE AND APPLICATIONS

For simplicity, we assume that given rings of characteristic p are localizations of finitely generated algebras over an excellent local ring (more generally, we may admit rings that have a faithfully flat extension such that the Frobenius endomorphism is finite). We assume that rings containing $\mathbb{Q}$ are locally excellent. These hypotheses are often unnecessary but avoid many technicalities. Let us call a Noetherian ring R *admissible* if it is equal characteristic and has the appropriate properties specified above.

In [HH12] a notion of tight closure is developed for the class of Noetherian K-algebras, where K is a given field of equal characteristic zero. This type of tight closure is denoted ^{*K} in [HH12].

We shall focus here on the notion that corresponds to the case $K = \mathbb{Q}$, which is also called *equational tight closure* in [HH12] and in is denoted there either by ${}^{*\mathbb{Q}}$ or ^{*eq}. However, here we shall often simply denote it *.

Tight closure is an operation on submodules of finitely generated modules defined over any admissible base ring. We shall denote the tight closure of N in M by N^* or ${N^*}_M$. If $N = I$ is an ideal of R then M is understood to be R unless otherwise specified: this is a very important case. If $N \subseteq M$ are R-modules and $R \to S$ is a homomorphism we shall write M_S for $S \otimes_R M$ and $\langle N_S \rangle$ for the image of N_S in M_S. This is an abuse of notation since $\langle N_S \rangle$ depends on $N \to M$ and not just on N. In the case when M is free, forming $\langle N_S \rangle$ is easily seen to be parallel to expanding an ideal of R to an ideal of S (which is the case $M = R$).

By a *complete local domain* of R we mean a ring obtained by localizing R at a prime ideal, completing, and then killing a minimal prime. We define the *minheight* of $I \subseteq R$ as $\min\{\mathrm{ht}\, I(R/P) : P \text{ is a minimal prime of } R\}$. We write $\overline{J}$ for the integral closure of the ideal J.

(2.1) Theorem (properties of tight closure). *Let R, S be admissible rings, $R \to S$ a homomorphism, and let $N \subseteq M \subseteq Q$ be finitely generated R-modules. Tight closures are taken in M unless otherwise specified. Let u denote an element of M, and I an ideal of R.*

(a) *$u \in {N^*}_M$ if and only if $u + N \in {0^*}_{M/N}$.*
(b) *${N^*}_M \subseteq {N^*}_Q$; ${N^*}_Q \subseteq {M^*}_Q$; $N^{**} = N^*$.*
(c) *(Persistence of *.) If $u \in N$ then $1 \otimes u \in {\langle N_S \rangle^*}_{M_S}$.*
(d) *$u \in N^*$ iff for every complete local domain B of R, $1 \otimes u \in {\langle N_B \rangle^*}_{M_B}$.*
(e) *If R is a regular ring, $N^* = N$ for all $N \subseteq M$.*
(f) *(Generalized Briançon-Skoda theorem) If I has at most n generators then for all $k \in \mathbb{N}$, $\overline{(I^{n+k})} \subseteq (I^{k+1})^*$.*
(g) *(Capturing colon ideals) If $I = (x_1, \ldots, x_n)R$ is such that $\mathrm{minheight}\, I \geq n$, and $I_{n-1} = (x_1, \ldots, x_{n-1})R$, then $I_{n-1} :_R x_n R \subseteq I^*_{n-1}$. Better: $I^*_{n-1} :_R x_n R = I^*_{n-1}$.*
(h) *(Phantom acyclicity criterion) Suppose that R is reduced, that $G_\bullet$ is a finite free complex of length n over R with rank $G_i = b_i$, and let $r_i = \sum_{j \geq i} (-1)^{j-i} b_j$. If $G_\bullet$ is such that the rank of the i^{th} map is r_i for $1 \leq i \leq n$ and such that the minheight of the ideal generated by the r_i size minors of a matrix of the map $G_i \to G_{i-1}$ is at least*

i, then for all $i \geq 1$ the cycles in G_i are in the tight closure of the boundaries. (We say that $G_\bullet$ is phantom acyclic in this case.)

(i) *If S is a module-finite extension of R then $u \in N^*{}_M$ over R if and only if $u \in \langle N_S \rangle^*{}_{M_S}$ over S. Moreover, the contraction of IS to R is contained in I.*

Proof. The reader is referred to [HH12] for the characteristic zero assertions, especially §(3.2), §(3.6) and, for (h), Chapter 4. For characteristic p see [HH4], §5 for (f), §8 for (a), (b), (e), [HH9], §6 for (c), [HH8], [AHH] for (h), [HH4] §7 for (g) and (5.23) of [HH10] for (i). Also, §§(1.4) - (1.6) in [HH12] give a summary of characteristic p theory. $\square$

(Two comments on part (h). First, it is easy to generalize the result to complexes consisting of projective modules that are locally free of constant rank. Second, in characteristic p, if $\mathbf{F}^e$ is the e^{th} iterate of the Frobenius functor then $\mathbf{F}^e(G_\bullet)$ is phantom acyclic for all e if and only if the specified conditions on the ranks and heights given above in (h) hold. See [HH8].)

Assuming these properties for tight closure one gets many striking results. Let us call a ring *weakly F-regular* if every ideal is tightly closed. Let us call a ring *F-rational* if every ideal I generated by parameters (i.e., elements that are part of a system of parameters in every local ring at a prime containing I) is tightly closed. It is not difficult to show that if R is weakly F-regular then every submodule of every finitely generated module is tightly closed. (In characteristic zero, one gets a notion for Noetherian K-algebras relative to every field K. Here, we are working with the notion for $K = \mathbb{Q}$.) R is called *F-regular* if all of its local rings are weakly F-regular. F-rational, weakly F-regular, and F-regular are known to be equivalent conditions if the ring is Gorenstein. Cf. [HH9], §4.

By the *regular closure* N^{reg} of $N \subseteq M$ we mean the set of all $u \in M$ such that for every map of R to a regular ring S, $u_S \in \langle N_S \rangle$ (in M_S).

(2.2) Corollaries. *Let R be an admissible ring of equal characteristic. Let $N \subseteq M$ be finitely generated R-modules and let I be an ideal of R.*

(a) *$N^* \subseteq N^{reg}$ (this is known to be strict: cf.* [HH8], §5*).*
(b) *$I^* \subseteq I^{reg} \subseteq \overline{I}$ (the latter since integral closure is tested by mapping to DVR's).*
(c) *Weakly F-regular rings are normal and Cohen-Macaulay. In fact, F-rational rings are normal and Cohen-Macaulay.*
(d) *A direct summand (or a pure subring) of a weakly F-regular ring is weakly F-regular.*
(e) *A direct summand (or pure subring) of a regular ring is Cohen-Macaulay.*
(f) *A weakly F-regular ring is a direct summand of every module-finite extension.*

Proof. (a) is immediate from (2.1c,e). The first inclusion in (b) follows from (a), and the second from the fact that $u \in \overline{I}$ iff $u \in IV$ for every homomorphism from R to a discrete valuation ring V. It follows from (b) and (2.1f) in the case $n = 1, k = 0$ that $I^* = \overline{I}$ if I is principal, and from this it follows easily that weakly F-regular rings are normal. The Cohen-Macaulay property is then immediate from (2.1g). (d) is immediate from the persistence of tight closure, and (e) follows at once from (c) and (d).

Part (f) follows from (i) and the results of [Ho4]: see §5 of [HH10] for details. (The converse is true for locally excellent Gorenstein rings of characteristic p: see §6 of [HH10]. On the other, hand every normal ring containing $\mathbb{Q}$ is a direct summand of every module-finite extension, so that the converse is quite false in equal characteristic zero.) Of course,

(f) may be viewed as a generalization of the fact that equicharacteristic regular rings are direct summands of all their module-finite extensions. □

Tight closure gives some rather important results in a more general form. See [BrS], [LS], [LT], and [Sk] for background for the Briançon-Skoda theorem. The result that pure subrings of regular rings are Cohen-Macaulay has aroused a great deal of interest: see [Bou], [HR1-2]. The original interest arose from studying the question, is the ring of invariants of a linearly reductive linear algebraic group over a field K acting K-rationally on a regular K-algebra Cohen-Macaulay.

The term "F-rational" is used because there is a strong connection between this notion and the property of having rational singularities. Let us say that an affine K-algebra R has $\mathcal{P}$*-type*, where $\mathcal{P}$ is a certain property (e.g., weakly F-regular, F-rational, F-regular) if it arises as $K \otimes_A R_A$, where A is a finitely generated $\mathbb{Z}$-subalgebra of K, R is a finitely generated A-algebra, and for all maximal ideals μ of A, with $\kappa = A/\mu$, $R_\kappa = \kappa \otimes_A R_A$ has property $\mathcal{P}$. K. E. Smith has shown [Sm1,3] that affine K-algebras of F-rational type have rational singularities. The converse is open (but true in the graded case in dimension two: cf. [Fe].) I. Aberbach has shown [Ab3] that an admissible ring of characteristic p is F-rational if and only if whenever M/N has finite projective dimension then ${N^*}_M = N$.

One also gets:

(2.3) Theorem (vanishing theorem for maps of Tor). *Let A be a regular equicharacteristic domain, let R be module-finite extension domain of A and let $R \to S$ be any map to a ring that is regular (or admissible and weakly F-regular). Then for any finitely generated A-module M and every $i \geq 1$, the map $Tor_i^A(M,R) \to Tor_i^A(M,S)$ is zero.*

Proof. Take a finite free resolution $G_\bullet$ of M over A. One sees easily that, since $G_\bullet$ satisfies the criterion of [BuE], $R \otimes_A G_\bullet$ satisfies the hypothesis of (2.1h), and so has phantom homology ($A \to R$ does not preserve depth but does preserve height). By the persistence of tight closure the image z of a cycle in $S \otimes_A G_\bullet$ is still in the tight closure of the boundaries. But since S is weakly F-regular, z is now a boundary. □

This extremely powerful vanishing theorem becomes a very interesting conjecture in mixed characteristic. The case where R is local and S is the residue field of R is equivalent to the direct summand conjecture. The case where R is a direct summand of S easily yields another proof that direct summands of regular rings are Cohen-Macaulay. See §4 of [HH11] for a discussion. (Cf. [PS1-2], [Ho1-3, 5-7], [Ro1-5], [Du] and [EvG] for background on these and other related questions.)

3. WHAT IS TIGHT CLOSURE IN CHARACTERISTIC p?

We next give the definition of tight closure in characteristic p. If R is a ring, let R^o be the set of elements of R not in any minimal prime. If I is an ideal of R, we say that an element $u \in R$ is in the tight closure I^* of I if there exists $c \in R^o$ such that $cu^{p^e} \in I^{[p^e]}$ for all sufficiently large $e \in \mathbb{N}$. Here, $I^{[p^e]}$ is generated by all the elements i^{p^e} for $i \in I$. If we replace R by a free module M and I by a submodule N, we can make the same definition: the $(p^e)^{th}$ power of an element of M may be defined by taking $(p^e)^{th}$ powers in each coordinate. In the general case, one may map a finitely generated free module onto M and work with it and inverse images for the submodule N and the element u in the free module. The notion one gets is independent of how one chooses a free basis and of how one maps a free module onto M. There is a coordinate-free treatment making use of the Peskine-Szpiro or Frobenius functors of [PS1].

It turns out that one may test tight closure modulo minimal primes of R. In the reduced case, one can choose the multiplier c so that $cu^{p^e} \in N^{[p^e]}$ for all $e \in \mathbb{N}$ (even $e = 0$). In the admissible case one can test tight closure in all the complete local domains of R, i.e., in all the rings obtained by localizing, completing, and killing a minimal prime.

Since this is the case it is of considerable interest to have alternative characterizations of tight closure when R is a complete local domain (several of those stated below are, in fact, valid under much weaker hypotheses). We give some of these characterizations in Theorem (3.1): we shall discuss additional characterizations later. For simplicity we only treat the case of the tight closure of an ideal of R: in all instances, there is an analogous result for modules.

For a reduced ring R of characteristic p we shall write R^∞ for $\bigcup_{e\in\mathbb{N}} R^{1/p^e}$. For a domain R, by R^+ , the *absolute integral closure* of R (cf. [Ar2]), we mean the integral closure of R in an algebraic closure of its fraction field.

(3.1) Theorem. *Let (R, m, K) be a complete local domain of characteristic p. Let I be an ideal of R and let $u \in R$. Then each of the following conditions, sometimes in the presence of a supplementary hypothesis, is equivalent to the condition that u be in I^*.*

(a) *Fix a discrete $\mathbb{Z}$-valuation nonnegative on R and positive on m and extend it to a valuation of R^+ to $\mathbb{Q}$. Then there exist elements $\epsilon \in R^+ - \{0\}$ of arbitrarily small order (i.e., value under the valuation) such that $\epsilon u \in IR^+$!*

(b) *Assume that I is m-primary. With $J = I + uR$, we have that*

$$\lim_{e\to\infty} \frac{\ell(R/J^{[p^e]})}{\ell(R/I^{[p^e]})} = 1 \; !$$

(Here, "ℓ" indicates length.)

(c) *(K. E. Smith* [Sm1,2]*.) Assume that I is generated by part of a system of parameters. Then $u \in IR^+$!*

(d) *There exists a big Cohen-Macaulay algebra S for R such that $u \in IS$!*

(e) *The element u is in the closure of I in the formal power series sense described in the Prologue*!

Proof and remarks. By and large we just supply references, but it is easy to see that $u \in I^*$ implies (a), for if we have $c \neq 0$ such that $cu^{p^e} \in I^{[p^e]}$ for all $e \gg 0$, then taking $(p^e)^{th}$ roots yields that $(c^{1/p^e})u \in IR^{1/p^e} \subseteq IR^\infty \subseteq IR^+$, and as $e \to +\infty$, $\operatorname{ord}(c^{1/p^e}) = (1/p^e) \operatorname{ord} c \to 0$. The converse is one of the main results of [HH6].

Part (b) is a consequence of Theorem (8.17) of [HH4] and the results of [Mo]: see the discussion below. We want to comment here that this result really does characterize tight closure in complete local domains of characteristic p, since an element is in the tight closure of $I \subseteq m$ if and only if it is in the tight closure of all m-primary ideals containing I.

Part (c) was proved for ideals generated by at most three parameters in [HH11] and in general in [Sm1,2] by a different method. Whether tight closure in locally excellent domains of characteristic p is, in general, simply the contracted expansion from R^+ is a tantalizing question. It does reduce to the case of complete local domains. (One knows that the contracted expansion from R^+ is contained in the tight closure by (2.1i).) This question is open even in dimension 2.

Part (d) and the peculiar fact (e) are results that are obtained in [Ho8] from the perspective of solid closure: we shall discuss them again in §5 and §6 (cf. (5.6) and (6.7)). □

Let us write $q = p^e$. When I is primary to m, $\ell(R/I^{[q]})$ as a function of e is called the *Hilbert-Kunz function* (cf. [Ku], [Mo], [HaMo]). (Again, "ℓ" denotes length.) By a result of [Mo], if $\dim R = d$ there is a positive real number C_I such that for all $q = p^e$, $\ell(R/I^{[q]}) = C_I q^d + O(q^{d-1})$. If $\dim R = 1$ then C_I is a positive integer. In general, C_I is conjectured to be rational, but this is an open question even in dimension two! By Theorem (8.17) of [HH4], when $I \subseteq J$ are m-primary we have that $J \subseteq I^*$ iff

$$\lim_{e\to\infty} \frac{\ell(J^{[q]}/I^{[q]})}{q^d} = 0,$$

which, by Monsky's result, is equivalent to the condition that $C_I = C_J$. This in turn is equivalent to condition (b) above.

The behavior of the Hilbert-Kunz functions is quite surprising. For example, if $R = (\mathbb{Z}/5\mathbb{Z})[[x_1, x_2, x_3, x_4]]/(G)$ where $G = \sum_{i=1}^4 x_i^4$, then

$$\ell(R/m^{[5^e]}) = \frac{168}{61}(5^e)^3 - \frac{107}{61}(3^e).$$

Cf. [HaMo]. See also [Ch].

We want to comment briefly on the theory of strongly F-regular rings and the theory of test elements in characteristic p. A reduced Noetherian ring R of characteristic p is called *strongly F-regular* if R is F-finite (i.e., R is module-finite over the image $F(R) = R^p$ of the Frobenius endomorphism) and for every $d \in R^o$ there exists $q = p^e$ such that (equivalently, for all sufficiently large $q = p^e$) the inclusion of the cyclic R-module $R \cdot d^{1/q} \hookrightarrow R^{1/q}$ splits as a map of R-modules. F-finite Gorenstein rings are strongly F-regular if and only if they are weakly F-regular. It is an open question whether an F-finite weakly F-regular ring must be strongly F-regular: cf. [Wil], where this is established in dimension less than or equal to three.

J. Velez has developed an analogous theory for F-rational rings in characteristic p. See [Vel].

An element $c \in R^o$ is called a *test element* if whenever $u \in N^*{}_M$ for some pair of finitely generated modules $N \subseteq M$ (one may assume that M is free), $cu^{p^e} \in N^{[p^e]}$ for all $e \in \mathbb{N}$. Thus, c may be used in all tight closure tests. A crucial point in the development of the theory is that if R is F-finite, reduced, and $c \in R^o$ is such that R_c is strongly F-regular (in particular, if R_c is regular) then c has a power that is a test element in R, in every local ring of R, and in the completion of every local ring of R. A test element satisfying this stronger condition is called *completely stable*. See §3 of [HH3] and §5 of [HH9]. In consequence, one can show that if R is admissible and reduced then every element $c \in R^o$ such that R_c is regular has a power that is a completely stable test element.

It is natural to study ideals generated by R-sequences that consist of test elements. The issue arises in trying to prove the results of §6 of [HH10]. It was in this context that it was first realized that:

(3.2) Theorem. *Let I be an ideal of a local domain R such that I is generated by the elements of a regular sequence consisting of test elements. Then $IR^+ \cap R = I^*$.*

(There is now a much better result: see (3.1c) and [Sm1,2].) The proof depended on an Equational Lemma that was eventually published in [HH7], where the circle of ideas that had arisen in [HH10] was used to prove the existence of big Cohen-Macaulay algebras. We shall return to this subject in §5, but we first explain how to define tight closure in equal characteristic 0.

4. WHAT IS TIGHT CLOSURE IN EQUAL CHARACTERISTIC ZERO?

We begin by discussing a family of characteristic p examples. Suppose that we let $R = K[X,Y,Z]/(X^3+Y^3+Z^3) = K[x,y,z]$, where K is a field of positive characteristic p. We assume that $p \neq 3$. We want to observe that z^2 is in $(x,y)^*$ regardless of the value of p. In fact, $x(z^2)^q \in (x,y)^{[q]} = (x^q,y^q)$ for all q, because if $2q = 3k+r$ where $r \in \{1,2\}$ we can rewrite xz^{2q} as $\pm x(x^3+y^3)^k z^r$ and it turns out that $xx^{3i}y^{3(k-i)} \in (x^q,y^q)$ for all i, $0 \le i \le k$, since $(3i+1)+(3k-3i) = 3k+1 \ge 2q-1$, so that one of the exponents is at least q. This is an interesting example, because it turns out that z is not in the tight closure of (x,y) but it is in the regular closure (and, of course, in the integral closure). See §5 of [HH8].

We shall adopt the following conventions: If R_A is an A-algebra, and M_A is an A-module, u_A is in R_A or M_A, and $A \to B$ is a homomorphism, we write R_B, M_B, u_B for $B \otimes_A R_A$, $B \otimes_A M_A$ and $1 \otimes u_A$ (in R_B or M_B, as the case may be), respectively. If $N_A \subseteq M_A$ we write $\langle N_B \rangle$ for $\mathrm{Im}\,(N_B \to M_B)$.

Now consider the $\mathbb{Z}$-algebra $R_{\mathbb{Z}} = \mathbb{Z}[X,Y,Z]/(X^3+Y^3+Z^3) = \mathbb{Z}[x,y,z]$. If we imagine tight closure to be a geometric notion, then $z^2/1 \in R_{\mathbb{Q}}$ ought to be in the tight closure of $(x,y)R_{\mathbb{Q}}$ because it is true that the image of z^2 in $R_{\mathbb{Z}/p\mathbb{Z}}$ is in the tight closure of $(x,y)R_{\mathbb{Z}/p\mathbb{Z}}$ for almost all the closed fibers ("almost all" means on a Zariski dense open subset of $\mathrm{Max\,Spec}\,\mathbb{Z}$ here): what happens at the generic fiber should be governed by what happens on a Zariski dense open subset of the closed fibers.

This is exactly the viewpoint we take to define K-tight closure over a field K of characteristic zero. Suppose that we are given a domain A contained in K and finitely generated over $\mathbb{Z}$, a ring R_A finitely generated over A, a pair of modules $N_A \subseteq M_A$, and an element $u_A \in M_A$. If $u_\kappa \in {N_\kappa}^*{}_{M_\kappa}$ for almost all maximal ideals μ of A ($\kappa = A/\mu$ will be a finite field), we decree that $u_K \in {N_K}^*{}_{M_K}$ over R_K . We require this not only for R_K: we also require that the image of u_K in $S \otimes_{R_K} M_K$ be in the tight closure, over S, of the image of $S \otimes_{R_K} N_K$ in $S \otimes_{R_K} M_K$ for all Noetherian K-algebras S to which R_K maps.

We note that the results of [Ar1], [ArR], and [Rot] play an important role in establishing that this notion has the right properties. (When R is not necessarily admissible, we define

u_R to be in ${N_R}^*{}_{M_R}$ if u_B is in $\langle N_B \rangle^*_{M_B}$ in the sense above for all complete local domains B of R.)

The theory we have been using here is $*\mathbb{Q}$. We do not know whether $*K$ and $*L$ really are different when $K \subseteq L$: quite generally, $N^{*K}{}_M \subseteq N^{*L}{}_M$. There is yet another notion, $*EQ$, which we may describe briefly as follows: let R be a local ring of a finitely generated $\mathbb{Q}$-algebra, and write R as C_P where C is a finitely generated $\mathbb{Z}$-algebra and P is a prime disjoint from $\mathbb{Z} - \{0\}$. Let $N_C \subseteq M_C$ be finitely generated C-modules and $u_C \in M_C$. Let $B(p)$ be the localization of C/pC at the multiplicative system of nonzerodivisors in C/pC on $C/(P + pC)$. Then $u_R \in {N_R}^{*EQ}{}_{M_R}$ if for almost all p, $u_{B(p)} \in {M_{B(p)}}^*{}_{N_{B(p)}}$.

More generally, if S is admissible, we define $u \in N^{*EQ}{}_M$ if for every prime Q of S, $(S_Q, M_Q, N_Q, u/1)$ may be obtained as $S \otimes_R (R, M_R, N_R, u_R)$ for some homomorphism $R \to S$, where R is as above and $u_R \in {N_R}^{*EQ}{}_{M_R}$ in the sense defined in the preceding paragraph.

It turns out that if S is a Noetherian K-algebra, where K is a field of characteristic 0, then $N^{\mathbb{Q}} \subseteq N^{*K} \subseteq N^{*EQ}$ when $N \subseteq M$ are finitely generated S-modules. We do not know whether $N^{*\mathbb{Q}} = N^{*EQ}$.

5. BIG COHEN-MACAULAY ALGEBRAS

We recall that M is a *balanced big Cohen-Macaulay module* for a local ring (R, m) if every system of parameters for R is a regular sequence on M. (The term *balanced* is used in [Sh].) This implies that $mM \neq M$. Henceforth we omit the term "balanced": by a *big Cohen-Macaulay module* we always mean a balanced big Cohen-Macaulay module. It may be the case that a big Cohen-Macaulay module M is also an R-algebra, in which case we shall say that M is a *big Cohen-Macaulay algebra* for R.

As noted in the last paragraph of §3, the investigation of tight closures of parameter ideals eventually lead to the following theorem, which is the main result of [HH7]. (Note: R^+ is defined in the discussion preceding (3.1).)

(5.1) Theorem. *Let R be an excellent local domain of characteristic p. Then R^+ is a big Cohen-Macaulay algebra for R.*

This is false in equal characteristic zero if the Krull dimension of R is at least three!

The applications of (5.1) are numerous. One gets a new equicharacteristic proof of the Faltings connectedness theorem (cf. (6.8) of [HH7]). The result is used in [Ab1] to show that if a finite free complex $G_\bullet$ over a reduced admissible local ring of characteristic p satisfies the conditions of (2.1h) and is minimal (i.e., the entries of the matrices of the maps are in the maximal ideal) then the ranks of the free modules occurring depend only on what $H_0(G_\bullet)$ is (the complex is referred to as a *finite phantom free resolution* of $H_0(G_\bullet)$). Note that the isomorphism class of such a complex is not uniquely determined by $H_0(G_\bullet)$. See [Ab1]. (5.1) also implies that locally excellent rings of characteristic p that are direct summands of all their module-finite extensions are Cohen-Macaulay (this may actually characterize F-regularity: it does for locally excellent Gorenstein rings, by Theorem (6.7) of [HH10]): see §7 of [HH7]. Moreover, a graded version of (5.1) can be used to bound the numbers of generators of certain homogeneous primes (cf. §8 of [HH11]) and to give some rather odd constraints on pairs of systems of parameters (cf. §9 of [HH11]).

But the point that we really want to focus on here is that (5.1) gives a weakly functorial method of obtaining big Cohen-Macaulay algebras: if we have any map of domains $R \to S$, one can extend it (not uniquely) to a map $R^+ \to S^+$. To see this, factor the map as the composition of a surjection $R \twoheadrightarrow \operatorname{Im} R$ and an inclusion $\operatorname{Im} R \hookrightarrow S$. It suffices to solve the problem for each map separately. In the case of an inclusion, $R \hookrightarrow S$, note that an algebraic closure of the fraction field of S will contain an algebraic closure of the fraction field of R, which gives the result at once. In the case of a surjection, we may assume that $S = R/P$ with P prime. Since R^+ is integral over R it has a prime ideal Q lying over P. Then $S = R/P$ injects into R^+/Q, and R^+/Q is an integral extension of S such that every monic polynomial factors into linear factors. This implies that R^+/Q may be identified with S^+.

Call a local homomorphism of local rings $R \to S$ *permissible* if the induced homomorphism $h : \hat{R} \to \hat{S}$ has the property that for every minimal prime $\mathfrak{q}$ of $\hat{S}$ such that $\dim \hat{S}/\mathfrak{q} = \dim \hat{S}$, there is a minimal prime $\mathfrak{p}$ of $\hat{R}$ such that $\mathfrak{p} \subseteq h^{-1}(\mathfrak{q})$ and $\dim \hat{R}/\mathfrak{p} = \dim \hat{R}$. One of the main results of [HH11] uses the weak functoriality of R^+, Theorem (5.1), and a rather non-routine application of Artin approximation [Ar1] to show the following:

(5.2) Theorem. *One may assign to every equicharacteristic local ring R a big Cohen-Macaulay algebra $\mathcal{B}(R)$ in such a way that whenever $R \to S$ is a permissible local homomorphism it extends to a homomorphism $\mathcal{B}(R) \to \mathcal{B}(S)$.*

This yields at once a proof that direct summands (or pure subrings) of regular rings are Cohen-Macaulay. Recall that $R \to S$ is *pure* if $R \otimes_R M \to S \otimes_R M$ is injective for every R-module M. This implies that for every ideal I of R, the contraction of IS to R is I.

(5.3) Corollary. *If S is an equicharacteristic regular ring and R is pure in S, then R is Cohen-Macaulay.*

Sketch of proof. One can reduce easily to the case where $R \to S$ is local. Since S is regular, $\mathcal{B}(S)$ is faithfully flat over S, which implies that S is pure in $\mathcal{B}(S)$. Then R is pure in $\mathcal{B}(S)$. Since we have $R \to \mathcal{B}(R) \to \mathcal{B}(S)$ this implies that R is pure in $\mathcal{B}(R)$. Since a system of parameters for R is a regular sequence in $\mathcal{B}(R)$, it is a regular sequence in R. □

Similarly:

(5.4) Theorem. *If R is a domain module-finite over an equicharacteristic regular ring A and $R \to S$ is a homomorphism to a regular ring, then for every finitely generated A-module M the map $\mathrm{Tor}_i^A(M, R) \to \mathrm{Tor}_i^A(M, S)$ is zero for all $i \geq 1$.*

Sketch of proof. (Cf. Theorem (4.1) of [HH11].) One reduces to the case where the rings are local. Let τ indicate $\mathrm{Tor}_i^A(M, \_)$. Then we have a factorization $\tau(R) \to \tau(S) \to \tau(\mathcal{B}(S))$. Since S is pure in $\mathcal{B}(S)$ (as before) it suffices to show that the composite is zero. But the map $\tau(R) \to \tau(\mathcal{B}(S))$ also factors $\tau(R) \to \tau(\mathcal{B}(R)) \to \tau(\mathcal{B}(S))$. Since $\mathcal{B}(R)$ is a big Cohen-Macaulay module for R, it is also a big Cohen-Macaulay module for A, and so flat over A. It follows that $\tau(\mathcal{B}(R)) = 0$. □

This result does not really depend on the fact that S is regular. It suffices to know that after localizing S at a maximal ideal, the ring one obtains is pure in a big Cohen-Macaulay algebra to which a big Cohen-Macaulay algebra for R can be mapped.

To make this idea precise, we define a 0^{th} *level big Cohen-Macaulay algebra* for an equicharacteristic local ring S to be the same as a big Cohen-Macaulay algebra. We define an $(n+1)^{th}$ *level big Cohen-Macaulay algebra* C for S to be one such that every permissible local homomorphism $R \to S$ extends to a map from some n^{th} level big Cohen-Macaulay algebra for R to C. We define C to be an ω^{th} *level big Cohen-Macaulay algebra* for S if it is an n^{th} level big Cohen-Macaulay algebra for S for all $n \in \mathbb{N}$. We define an equicharacteristic Noetherian ring S to be *weakly CM*^{n}*-regular*, where $n \in \mathbb{N} \cup \omega$, if each of its local rings at a maximal ideal is pure in an n^{th} level big Cohen-Macaulay algebra. If $n = 0$ this just means that S is Cohen-Macaulay. For $n \geq 1$ this notion is not transparent but we do have:

(5.5) Theorem. *The following results hold for equicharacteristic Noetherian rings:*

(a) *Theorem (5.4) remains valid if S, instead of being assumed regular, is instead assumed to be pure in a first or higher level big Cohen-Macaulay algebra for S.*
(b) *A weakly CM*^{n}*-regular ring for $n \geq 1$ is normal (and Cohen-Macaulay).*
(c) *An equicharacteristic local ring R is weakly CM*^{n}*-regular if and only if it is pure in $\mathcal{B}(R)$.*
(d) *If R is a locally excellent ring of characteristic p and R is weakly F-regular then R is CM*$^\omega$*-regular.*
(e) *If R is an affine K-algebra in which every ideal is tightly closed in the sense of $*^K$, then R is weakly CM*$^\omega$*-regular.*

Comments on the proof and references. For (a), the proof of Theorem (5.4) goes through without essential change. See Theorem (4.12) of [HH11]. Part (b) is Theorem (4.9d) of [HH11]. Part (c) is Proposition (5.5.e) of [HH11]. Parts (d) and (e) are parts (g) and (h) of Theorem (4.9) of [HH11]. □

Further remarks. We want to emphasize here that once one realizes that there are two similar vanishing theorems for maps of Tor, (5.5a) here and Theorem (2.3), one in which S is assumed to be admissible and weakly F-regular, the other in which S is assumed to be CM^n-regular for some $n \geq 1$, it is obvious that there ought to be some connection between the two properties. What is more, one starts to suspect that tight closure may be a contracted expansion from some sort of big Cohen-Macaulay algebra! Parts (d) and (e) above provide some evidence in this direction. Further evidence is provided by the following result:

(5.6) Theorem. *Let R be an excellent equicharacteristic local ring. Let $N \subseteq M$ be finitely generated R-modules and let $u \in M$.*

(a) *Suppose that $\hat{R}$ is equidimensional and $\mathcal{B} = \mathcal{B}(R)$. If $u_{\mathcal{B}} \in \langle N_{\mathcal{B}} \rangle$ then $u \in N^*{}_M$.*
(b) *If R has characteristic p and $\hat{R}$ is a domain, then $u \in N^*$ if and only if there exists a big Cohen-Macaulay algebra B for R such that $u_B \in \langle N_B \rangle$.*
(c) *If R is a complete local domain of characteristic 0 and $u \in N^{*EQ}{}_M$ (in particular, if $u \in N^*{}_M$) then there exists a big Cohen-Macaulay algebra B for R such that $u_B \in \langle N_B \rangle$. (See also Theorem (6.6).)*

Proof. Part (a) is part of Theorem (5.12) of [HH11]. Part (b), which has already been stated in a special case in Theorem (3.1d), is Theorem (11.1) of [Ho8]: it is proved from the perspective of solid closure. Part (c) is the main part of Theorem (11.4) of [Ho8]. All three arguments are lengthy and difficult. □

It is a challenging problem to refine these results connecting various kinds of tight closure with contracted expansions from big Cohen-Macaulay algebras. Notice that this circle of ideas offers the possibility of characterizing tight closure in equal characteristic zero in a way that does not make any reference to reduction to characteristic p.

Here is a tantalizing example.

(5.7) *Example.* Let $R = K[[X, Y, Z]]/(X^3 + Y^3 + Z^3) = K[[x, y, z]]$, where K is a field of characteristic different from 3. We have seen that in all cases, $z^2 \in (x, y)^*$. What we want to show here is that, as one might expect, in all cases, if B is any higher level Cohen-Macaulay algebra for R ("higher" meaning level at least 1) then $z^2 \in (x, y)B$. (5.6a) then implies, as we already know, that $z^2 \in (x, y)^*$. Thus, we are establishing the converse of (5.6a) in an example.

The trick is to take new variables s, t and form the subring

$$R_0 = K[[xs, ys, zs, xt, yt, zt]] \subseteq R[[s, t]].$$

There is a local K-homomorphism of R_0 to R that sends the formal generators xs, ys, zs, xt, yt, zt to x, y, z, x, y, z respectively. In R_0, xs, yt, $xt+ys$ is part of a system of parameters: call these elements u, v, w. One has the relation $(zs)(zt)(w) = (zt)(zt)u + (zs)(zs)v$. It follows that in any big Cohen-Macaulay algebra B_0 for R_0, $(zs)(zt) \in (xs, yt)B_0$. But some such B_0 maps to the first level or higher big Cohen-Macaulay algebra B for R, and so $z^2 \in (x, y)B$, as claimed. □

We conclude this section with the following closely related result, which does not appear elsewhere.

(5.8) Theorem. *Let (R, m, K) be a local ring, let I be an ideal of R, and let $u \in R$. Suppose that there exists an integral extension T of R such that $u \in IT$. Then for any first or higher level big Cohen-Macaulay algebra S for R, $u \in IS$.*

In particular, if R is a domain, then if $u \in IR^+$ it follows that $u \in IS$ for every higher level big Cohen-Macaulay algebra S for R.

Proof. Let Λ denote either a field contained in R or else the localization of $\mathbb{Z}$ at the prime $p\mathbb{Z}$, where p is the residual characteristic of R. In either case, there is an obvious local homomorphism $\Lambda \to R$. Let $a_1, \ldots, a_n$ denote generators for I, and let $x_1, \ldots, x_n$ be indeterminates over Λ. Write $u = \sum_{i=1}^n a_i\theta_i$, where every θ_i is an element of T (hence, integral over R). For every i choose an equation of integral dependence

$$g_i(z_i) = z_i^{\delta(i)} + \sum_{j=0}^{\delta(i)-1} r_{ij} z_i^j = 0$$

for θ_i over R, where the $r_{ij} \in R$. Let v_{ij} be new indeterminates, and let

$$G_i(z_i) = z_i^{\delta(i)} + \sum_{j=0}^{\delta(i)-1} v_{ij} z_i^j$$

in the polynomial ring $W = \Lambda[x_i, z_i, v_{ij}, t]$, where t is yet another indeterminate. There is a homomorphism $W \to T$ extending the map $\Lambda \to R$, and which sends the x_i to the a_i, the v_{ij} to the r_{ij}, the z_i to the θ_i, and t to 0. This homomorphism kills all the G_i, and so induces a homomorphism of $V = W/(G_i : i)$ to T. Let $y = \sum_{i=1}^n z_i x_i$. Note that y maps to u under $W \to T$. The subring $U = \Lambda[x_i, y, v_{ij}, t, z_i t]$ of V therefore maps to R, and, consequently, so does $U_{\mathcal{N}}$, where $\mathcal{N}$ is the contraction of the maximal ideal of R to U.

Let $B = \Lambda[x_i, y_{ij}, t]$, which is regular, let $\mathcal{M}$ denote the contraction of the maximal ideal of R to B, which contains the x_i and t, and let $C = B_{\mathcal{M}}$, which is a regular local ring. Then V is module-finite over B, and the monomials $z_1^{\nu_1} \cdots z_n^{\nu_n}$ in the z's such that $\nu_i < \delta(i)$ for all i are a free basis for V over B. Since $B \subseteq U \subseteq V$, the ring U is module-finite and torsion-free over B. It follows that $U_{\mathcal{N}}$ is a localization of $U_{\mathcal{M}}$ (which is semilocal) at an ideal lying over $\mathcal{M}C$. Here, $U_{\mathcal{M}}$ is module-finite and torsion-free over the regular local ring C. The ideals of $U_{\mathcal{M}}$ lying over $\mathcal{M}C$ are precisely the maximal ideals of $U_{\mathcal{M}}$, and since C is normal and $U_{\mathcal{M}}$ is torsion-free over C, we have going-down as well as going-up holding for the inclusion $C \hookrightarrow U_{\mathcal{M}}$. It follows that all the maximal ideals of $U_{\mathcal{M}}$ have the same height, equal to the dimension of C. Moreover, it follows that the image of any system of parameters for C is a system of parameters in each of the local rings of $U_{\mathcal{M}}$ at a maximal ideal, and, hence, in particular, in $U_{\mathcal{N}}$. Since $x_1, \dots, x_n, t$ is a regular sequence of B contained in $\mathcal{M}$, it is part of a system of parameters for C and, hence, its image in $U_{\mathcal{N}}$ is part of a system of parameters for $U_{\mathcal{N}}$.

Now, the completion of $U_{\mathcal{M}}$ with respect to $\mathcal{M}C$ will be the product of the completions of its local rings at its maximal ideals, one of which will be $(U_{\mathcal{N}})\hat{}$. Since $U_{\mathcal{M}}$ is torsion-free as a C-module, so is its $(\mathcal{M}C)$-adic completion $\mathcal{U}$ as a $\hat{C}$-module, and it follows that every nonzero $\hat{C}$-submodule of $\mathcal{U}$ has the same dimension as $\hat{C}$. This shows that $(U_{\mathcal{N}})\hat{}$ is equidimensional (and has no embedded primes). It follows that $U_{\mathcal{N}} \to R$ is permissible, and we have already seen that $x_1, \dots, x_n, t$ is part of a system of parameters in $U_{\mathcal{N}}$.

Since S is a first level big Cohen-Macaulay algebra for R, there exists a big Cohen-Macaulay algebra $\mathcal{B}$ for $U_{\mathcal{N}}$ such that $U_{\mathcal{N}} \to R \to S$ also factors $U_{\mathcal{N}} \to \mathcal{B} \to S$. In $U_{\mathcal{N}}$ we have the relation $yt = \sum_{i=1}^n x_i(z_i t)$, by the definition of $y = \sum_{i=1}^n x_i z_i$. But then, since $x_1, \dots, x_n, t$ is part of a system of parameters in $U_{\mathcal{N}}$, we must have that $y = \sum_{i=1}^n \beta_i x_i$ in $\mathcal{B}$. When we then map further to S, we obtain that the image of u in S is in $(a_1, \dots, a_n)S = IS$, as required, since, under the map $U_{\mathcal{N}} \to R$, y maps to u and x_i to a_i for all i. $\square$

(5.9) *Remark.* There is a corresponding result for modules instead of ideals. One can reduce to looking at submodules of finitely generated free modules, where the statement and proof are entirely similar.

(5.10) *Concluding discussion.* For simplicity, we assume that (R, m, K) is a complete local domain of equal characteristic and consider only the case of ideals. Let I be an ideal of R and let $u \in R$.

In characteristic p, we know that $u \in I^*$ if and only u is in the expansion of I to a big Cohen-Macaulay algebra for R. However, it need not be in the expansion of I to every big Cohen-Macaulay algebra: R itself can be Cohen-Macaulay without having the property that every ideal is tightly closed. Theorem (5.8) suggests the possibility that $u \in I$ if and only if it is in the expansion of I to every first or higher level big Cohen-Macaulay algebra for R. This will be the case if and only if it turns out that in characteristic p, $I^* = IR^+ \cap R$. Indeed, since R^+ is such an algebra, this statement forces I^* to be contained in $IR^+ \cap R$. On the other hand, Theorem (5.8) shows every element of $IR^+ \cap R$ is forced into the expansion of I to every first or higher level big Cohen-Macaulay algebra.

It is possible that in equal characteristic zero, I^{*eq} is simply what is in the contracted expansion of I from every first or higher level big Cohen-Macaulay algebra. Theorem (5.12) of [HH11] implies that any element that is in such a contracted expansion for all first or higher level big Cohen-Macaulay algebras is in I^{*eq}. (It actually shows this for a specific ω level big Cohen-Macaulay algebra. This is also stated as Theorem (5.6a) here.) Moreover, Theorem (11.4) of [Ho8] shows, at least, that every element of I^{*eq} (and even of I^{*EQ}) is in the expansion of I to some big Cohen-Macaulay algebra: what we don't know is whether requiring the algebra to be first or higher level automatically ensures that it is "big enough." (This is stated as Theorem (5.6c) here.) Example (5.7) simply presents one instance in which we can verify that this is, indeed, the case.

6. SOLID CLOSURE

In this section we shall discuss a closure operation defined in a characteristic-free manner that coincides with tight closure for admissible rings of characteristic p.

(6.1) *Solid modules and algebras: some basic properties.* If R is a domain we call an R-module M *solid* if $\mathrm{Hom}_R(M, R) \neq 0$. We call an R-algebra S *solid* if it is solid as an R-module. In the case of a solid algebra it is always possible to choose a nonzero R-module map $h : S \to R$ such that $h(1) \neq 0$ ([Ho8], Proposition (2.1d)). One reason for the name "solid" is that solid algebras tend not to adjoin fractions: if S is a solid R-algebra, $a, b \in R$, $b \neq 0$, and $a \in bS$, then a/b is integral over R; if θ is an element of a solid R-algebra and θ is algebraic over R, then θ is integral over R ([Ho8], Proposition (2.9)).

Moreover, if $R \to S$ is any homomorphism of domains with R Noetherian and M is a solid R-module then $S \otimes_R M$ is a solid S-module ([Ho8], Theorem (2.12)).

Also, it turns out that a module S over a complete local domain (R, m) of dimension d is solid if and only if the highest local cohomology $H^d_m(S) \neq 0$ (cf. [Ho8], Corollary (2.4)).

In consequence, a big Cohen-Macaulay algebra over a complete local domain is always solid. On the other hand, so is any module-finite extension algebra.

(6.2) *Solid closure.* Let R be a Noetherian ring, let $N \subseteq M$ be finitely generated R-modules, and let $u \in M$. We shall say that $u \in N^\bigstar$ (or $N^\bigstar{}_M$ if greater precision is needed), the *solid closure* of N in M, if for every complete local domain B of R, there exists a solid B-algebra S such that $u \in \langle N_S \rangle$. It suffices to consider complete local domains arising from completions at maximal ideals of R. (Note: the notation $N^\blacksquare$ was used for solid closure in a previous version of this manuscript.)

(6.3) *A connection with tight closure.* It is easy to see that if R is a Noetherian domain of characteristic p, I is an ideal, and $u \in IS$ for a solid R-algebra S, where $u \in R$, then $u \in I^*$. For we may choose an R-module map $h : S \to R$ such that $h(1) = c \neq 0$, and the fact that $u \in IS$ implies that $u^q \in I^{[q]}S$ for all $q = p^e$ (we may apply Frobenius). Applying h to both sides yields that $cu^q \in I^{[q]}$ for all q. For complete local domains, the converse is true: see Theorem (8.6) of [Ho8]. From this one can deduce:

(6.4) Theorem. *If R is an admissible ring of characteristic p, $N \subseteq M$ are finitely generated R-modules, and $u \in M$, then $u \in N^*$ if and only if $u \in N^\bigstar$.*

Does the new notion coincide with one of the equal characteristic zero tight closure notions already discussed? The answer was not known until recently, when a calculation of Paul Roberts [Ro6] revealed that in the rings $K[[x_1, x_2, x_3]]$ and $K[x_1, x_2, x_3]$, where K is a field of equal characteristic zero, one has that $x_1^2 x_2^2 x_3^2 \in (x_1^3, x_2^3, x_3^3)^\bigstar$. This means that in equal characteristic zero, solid closure is "too big." See also (7.22-5) in [Ho8].

As for mixed characteristic, the verdict is not yet in concerning whether solid closure gives the "right" notion or, at least, a sufficiently good notion, although Roberts' example is discouraging. Note, however that (x_1^3, x_2^3, x_3^3) (and every other ideal generated by monomials in a polynomial ring over $\mathbb{Z}$) is solidly closed. Cf. (13.1), (13.3) and (13.5-7) of [Ho8].

Even if it should turn out that solid closure is too big in mixed characteristic, it is still much smaller than integral closure, and so a result that asserts that a certain ideal is in the solid closure of another is much sharper than the corresponding result for integral closure. It remains possible that in any regular local ring of mixed characteristic, every ideal is solidly closed. This is true in dimension two, but not known in dimension three or higher. (As already mentioned, it is true for regular rings of characteristic p and false for regular rings of equal characteristic zero of dimension 3.) Settling the issue for regular rings of mixed characteristic is a primary target.

We next want to record some of the properties of solid closure. We follow the format of Theorem (2.1), including the numbering of the parts.

(6.5) Theorem (properties of solid closure). *Let R, S be Noetherian rings, $R \to S$ a homomorphism, and let $N \subseteq M \subseteq Q$ be finitely generated R-modules. Solid closures are taken in M unless otherwise specified. u denotes an element of M, and I an ideal of R.*

(a) *$u \in N^\bigstar{}_M$ if and only if $u + N \in 0^\bigstar{}_{M/N}$.*

(b) *$N^\bigstar{}_M \subseteq N^\bigstar{}_Q$; $N^\bigstar{}_Q \subseteq M^\bigstar{}_Q$; $(N^\bigstar)^\bigstar = N$.*

(c) *(Persistence of $^\bigstar$.) If $u \in N^\bigstar$ then $1 \otimes u \in \langle N_S \rangle^\bigstar{}_{M_S}$.*

(d) *$u \in N^\bigstar$ iff for every complete local domain B of R, $1 \otimes u \in \langle N_B \rangle^\bigstar{}_{M_B}$.*

(e) *If R is a regular ring of dimension at most two or of characteristic p, $N^\bigstar = N$ for all $N \subseteq M$. (Question: is this true for regular local rings of mixed characteristic?)*

(f) *(Generalized Briançon-Skoda theorem) If R is equicharacteristic and I has at most n generators then for all $k \in \mathbb{N}$, $\overline{I^{n+k}} \subseteq (I^{k+1})^\bigstar$. (Question: is this true in all Noetherian rings?)*

(g) *(Capturing colon ideals) If R is an equicharacteristic and locally excellent ring, $I = (x_1, \ldots, x_n)R$ is such that minheight $I \geq n$, and $I_{n-1} = (x_1, \ldots, x_{n-1})R$, then we* *have $I_{n-1} :_R x_nR \subseteq I^{\bigstar}_{n-1}$. (Question: is this true for local rings in mixed characteristic and does one get the better result that $I^{\bigstar}_{n-1} :_R x_nR = I^{\bigstar}_{n-1}$ in mixed characteristic?)*

(h) *The analogue of* (2.1h) *is valid for locally excellent rings of equal characteristic with solid closure replacing tight closure.*

(i) *If S is a module-finite extension of R then $u \in N^{\bigstar}{}_M$ over R if and only if $u_S \in \langle N_S\rangle^{\bigstar}{}_{M_S}$ over S. Moreover, the contraction of IS to R is contained in $I^{\bigstar}$.*

Proof. For (a), (b), (c), (d) we refer the reader to (5.3) and (5.6) of [Ho8]. Part (e) is Theorem (7.20) of [Ho8], and (f), (g), (h) follow in characteristic p from corresponding facts for tight closure and Theorem (6.4). In equal characteristic 0 they follow from corresponding facts for tight closure and the comparison Theorem (6.6) below. (i) follows from (5.9b) of [Ho8]. □

Wherever we have asked a question in the statement of Theorem (6.5) there is work to be done. We now return to the problem of comparing solid closure in equal characteristic zero with tight closure. Our main result along these lines is Theorem (11.4) of [Ho8] (part of this was mentioned here earlier, as Theorem (5.6c)), which asserts:

(6.6) Theorem. *Let R be any Noetherian ring containing $\mathbb{Q}$. Let $N \subseteq M$ be finitely generated R-modules and let $u \in M$.*

(a) *If R is a complete local domain and $u \in N^{*EQ}{}_M$ (hence, if $u \in N^*{}_M$) then there is a big Cohen-Macaulay algebra B over R such that $u_B \in \langle N_B\rangle$.*

(b) *$N^{*EQ}{}_M \subseteq N^{\bigstar}{}_M$. Hence, $N^*{}_M \subseteq N^{\bigstar}{}_M$.*

Remarks on the proof. A big Cohen-Macaulay algebra over a complete local domain is automatically solid. Thus, (a) yields (b) at once when R is a complete local domain, and the general case of (b) reduces to the complete local domain case. The proof of (a) is lengthy. □

As mentioned earlier, the result of [Ro6] shows that, in general, the inclusion $N^{*EQ}{}_M \subseteq N^{\bigstar}{}_M$ is strict, since in equal characteristic zero every ideal of a regular ring is tightly closed in the sense of ^{*EQ}, but not necessarily solidly closed.

(6.7) *Back to the Prologue: a formal power series criterion.* Let $I = (a_1, \dots, a_n)$ be an ideal of a complete local domain R, and let a be any element of R. It is not hard to see that if there is any solid R-algebra S such that $a \in IS$, then it must be $S = R[X_1, \dots, X_n]/(F)$ where $F = a - a_1X_1 - \cdots - a_nX_n$, because S maps to any other algebra T such that $a \in IT$ and if S maps to T and T is solid then S is solid. We refer to S as a *generic forcing algebra*. We can study the R-homomorphisms $h : S \to R$ by considering instead the formal power series in auxiliary variables y_i obtained from h by letting

$$g_h = \sum_{\nu \in \mathbb{N}^n} h(X^\nu)y^\nu \in R[[y_1, \dots, y_n]],$$

where $h(X^\nu)$ represents the value of h on the image of $X_1^{\nu_1} \cdots X_n^{\nu_n}$ in S. Let f be the power series

$$ay_1 \cdots y_n - \sum_{i=1}^{n} a_iy_1 \cdots y_{i-1}y_{i+1} \cdots y_n$$

(labeled (#) in the Prologue). It turns out not to be difficult to see (cf. Theorem (9.3) of [Ho8]) that g_h corresponds to a homomorphism that kills the ideal (F) if and only if $g_h f$ is *special* in the sense that it does not contain a term in which all of the y_i occur with positive exponent. It then follows that $a \in I^\bigstar$ if and only if f has a nonzero multiple that is special. This gives yet another characterization of tight closure for complete local domains of characteristic p. The examples considered in the Prologue really ask whether $x_1^2x_2^2x_3^2$ is in the solid closure of (x_1^3, x_2^3, x_3^3) in various regular local rings in which x_1, x_2, x_3 is a regular system of parameters. The answer is not known for the first example ($R = V[[x_2, x_3]]$) mentioned in the Prologue, and is "no" (respectively, "yes") for $K[[x_1, x_2, x_3]]$ when the field K has positive characteristic (respectively, characteristic zero).

Although we have discussed generic forcing algebras and the power series criterion only for ideals, these do extend to pairs of modules. We refer the reader to §9 of [Ho8].

(6.8) *Examples in regular rings, and an application of the Briançon-Skoda theorem.* On the other hand, it is easy to see that $x_1x_2x_3$ is not in the solid closure of (x_1^2, x_2^2, x_3^2) in the ring $K[[x_1, x_2, x_3]]$ (where K has characteristic 0) using the persistence of solid closure. Simply map $K[[[x_1, x_2, x_3]] \to K[[x, y]]$ so as to send x_1, x_2, x_3 to $x + y$, $x - y$, and xy, respectively. If $x_1x_2x_3 \in (x_1^2, x_2^2, x_3^2)^\bigstar$ we find that $(x^2+y^2)(x^2-y^2)xy \in (x^4+y^4, x^2y^2)^\bigstar$ in $K[[x, y]]$, and since we know that every ideal is solidly closed in a regular ring of dimension two, it follows that if $x_1x_2x_3 \in (x_1^2, x_2^2, x_3^2)^\bigstar$ then $(x^2 + y^2)(x^2 - y^2)xy \in (x^4 + y^4, x^2y^2)$ in $K[[x, y]]$, which is false. What is being used here is that $f = x^2 + y^2$, $g = x^2 - y^2$, $h = xy$ gives an example where $fgh \notin (f^2, g^2, h^2)$ in $K[[x, y]]$ when K has characteristic zero. This answers the second part of the "Problem for thought" given at the end of the Prologue.

It is tempting to try to use the same method to show that $x_1^2x_2^2x_3^2 \notin (x_1^3, x_2^3, x_3^3)^\bigstar$. Thus, one winds up seeking f, g, h in a regular ring of dimension two such that $f^2g^2h^2 \notin (f^3, g^3, h^3)$. (This was the first part of the "Problem for thought" following the Prologue.) But this is impossible! I don't know an elementary proof, but one can use the Briançon-Skoda theorem to show that in any regular ring of dimension two, $f^2g^2h^2 \in (f^3, g^3, h^3)$. The issue is local. The ideal (f^3, g^3, h^3) will have a minimal reduction (extend the residue field if necessary: this will not affect any relevant issue) generated by at most two elements, say, (u, v). Now, $(fgh)^3 = f^3g^3h^3$ shows that fgh is integral over (f^3, g^3, h^3) and, hence, over (u, v), so that the Briançon-Skoda theorem implies that $(fgh)^2$ (which is in $\overline{(u, v)^2}$) is in $(u, v) \subseteq (f^3, g^3, h^3)$. (We are using the version of [LS], which is valid for all regular rings.) Similarly, in any regular ring of dimension at most d, the product of the d^{th} powers of any $d + 1$ elements is in the ideal generated $(d + 1)^{th}$ powers of those elements.

As mentioned earlier, whether $x_1^2x_2^2x_3^2 \in (x_1^3, x_2^3, x_3^3)^\bigstar$ for a regular system of parameters in a regular local ring of mixed characteristic remains open.

★ ★ ★

EPILOGUE: TWENTY QUESTIONS

We give here a list of twenty open questions, taken in part from a preliminary version of [HH12], connected with tight closure theory.

Throughout these questions, unless otherwise specified, R is a locally excellent Noetherian ring and $N \subseteq M$ are finitely generated R-modules. K, L always denote fields and $K \subseteq L$. If R is local then it has maximal ideal m and residue field K.

If $M = \operatorname{Coker}(a_{ij})$, $\mathbf{F}^e(M) = \operatorname{Coker}(a_{ij}^q)$ with $q = p^e$.

1. Does tight closure commute with localization in characteristic p? In characteristic zero? (Cf. [AHH] for a thorough discussion of the localization problem.)

2. Is it true that weakly F-regular rings are F-regular? This is known in the Gorenstein case (and in dimension at most three, using the results of [Wil]), but is open in characteristic p even if R is an affine algebra over an algebraically closed field.

3. If R is complete local, is there a positive constant integer b such that for all $e \in \mathbb{N}$, m^{bp^e} kills $H_m^0(\mathbf{F}^e(M))$? Is this true even when R is a complete local weakly F-regular ring and $M = R/I$, where I is primary to a prime P such that $\dim R/P = 1$? An affirmative answer to the second question would yield that weakly F-regular implies F-regular for locally excellent rings R. (Cf. [AHH].)

4. In characteristic p, is every F-finite weakly F-regular ring strongly F-regular? (Then weakly F-regular would imply F-regular even without the hypothesis F-finite.) This is known:
   (1) in the Gorenstein case and
   (2) if $\dim R \leq 3$ (cf. [Wil]).

5. Let R be a complete local Cohen-Macaulay domain and let J be an ideal of R that is isomorphic as a module with a canonical module for R. Fix a system of parameters $x_1, \ldots, x_n$ for R. For every $q = p^e$ and every positive integer t let

$$\theta_{e,t} : J^{[q]}/(x_1^q, \ldots, x_n^q) \to J^{[q]}/(x_1^{qt}, \ldots, x_n^{qt})$$

be the map induced by mutliplication by $(x_1 \cdots x_n)^{qt-q}$ on $J^{[q]}$. Can one always choose a system of parameters $x_1, \ldots, x_n$ and a positive integer t_0 such that $\operatorname{Ker} \theta_{e,t}$ is the same for all $t \geq t_0$? For fixed e, these kernels increase as t increases. Note that t_0 is to be independent of e. This and related problems are studied in [Wil]. An affirmative answer in the special case where R is F-finite and weakly F-regular would suffice to show that weakly F-regular is equivalent to strongly F-regular for F-finite rings, and that weakly F-regular is equivalent to F-regular for locally excellent rings of characteristic p.

6. Is the weakly F-regular locus open? (This is an open question for affine rings over algebraically closed fields both in characteristic p and characteristic 0.) This would follow for algebras essentially of finite type over an excellent local ring of characteristic p if weakly F-regular F-finite rings are strongly F-regular, because the strongly F-regular locus is known to be open.

7. Suppose that R has characteristic p. Let $T = \bigcup_e \operatorname{Ass}(\mathbf{F}^e(M)/0^*)$. Is T finite? Does it have only finitely many maximal elements? An affirmative answer would reduce the question of whether tight closure commutes with localization to the case where R is local and one is localizing at a prime P with $\dim R/P = 1$. (Cf. [AHH].)
(A recent example of M. Katzman shows that $S = \bigcup_e (\operatorname{Ass} \mathbf{F}^e(M))$ need not have only finitely many maximal elements. Katzman has also shown that there is a polynomial

ring C in one variable over a field of characteristic p, a finitely generated C-algebra R, and a finitely generated R-module M such that there is no element $c \in C^o$ such that all the modules $F^e_{R_c}(M_c)$ are torsion-free over C_c.)

8. Is characteristic p tight closure for locally excellent domains the same as contraction from R^+ ? This is known for parameter ideals ([Sm1,2]). An affirmative answer implies that tight closure commutes with localization. This is an open question even in dimension two.

9. Does weak F-regularity deform, i.e., if R is a local domain, $x \neq 0$ and R/xR is weakly F-regular must R be weakly F-regular? (This is known when R is Gorenstein.)

10. Are affine K-algebras with rational singularities necessarily of F-rational type? (The converse is true: [Sm1,3].) (This is known in the graded case in dimension two ([Fe]) but open in general even in dimension two.)

11. For affine algebras in characteristic 0, does weakly F-regular imply weakly F-regular type? Does F-rational imply F-rational type? Are affine algebras with rational singularities of F-rational type (the converse is known: [Sm1,3])?

12. Are direct summands (and pure subrings) of F-rational rings F-rational?

13. Let K have characteristic 0, let A be a finitely generated $\mathbb{Z}$-subalgebra of K, and let (A, R_A, M_A, N_A) be descent data for (K, R, M, N). Suppose that N is K-tightly closed in M. Is it true that for almost all closed fibers, N_κ is tightly closed in M_κ?

14. Is tight closure over a complete local domain, in the equicharacteristic case (0 and p), simply the contracted expansion from a higher level balanced big Cohen-Macaulay algebra?

15. If R is an affine L-algebra, does ${N^{*K}}_M = {N^{*L}}_M$? More generally, if R is an arbitrary Noetherian ring containing $\mathbb{Q}$ does ${N^{*eq}}_M = {N^{*EQ}}_M$? (It would suffice to know this when R is a local ring of a finitely generated $\mathbb{Q}$-algebra.)

16. If a flat homomorphism of rings $R \to S$ has an F-regular base R and geometrically F-regular fibers, is S F-regular? This is known in good cases in characteristic p if the fibers are geometrically regular. (Cf. [HH9].)

17. If R is excellent, reduced, characteristic p, and of finite Krull dimension, does R have a test element? (Cf. [Ab2].)

18. Under mild conditions on a characteristic p ring R (e.g., if R is reduced and finitely generated as an algebra over an excellent local ring) does formation of the ideal of test elements commute with localization? With completion? With geometrically regular base change (where *geometrically regular* means flat with geometrically regular fibers)?

19. Let R be a characteristic p local ring such that one system of parameters generates a tightly closed ideal. Is R F-rational? This is known ([HH9] §4) if R is equidimensional.

20. Is every ideal of a regular ring solidly closed in mixed characteristic? (The mixed characteristic case implies the direct summand conjecture.) The question raised in the prologue is really the question of whether (x_1^3, x_2^3, x_3^3) is solidly closed in $V[[x_2, x_3]]$. Every ideal is solidly closed in regular rings of dimension at most two, even in mixed characteristic.

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