Test elements using the Lipman-Sathaye theorem

Theorem. *Let $R \subseteq S$ be Noetherian domains of positive characteristic p with fraction fields $\mathcal{K} \subseteq \mathcal{L}$ such that R is regular, S is module-finite over R, and $\mathcal{L}$ is separable over $\mathcal{K}$. Then $R^{1/q} \otimes_R S \cong R^{1/q}[S]$ is faithfully flat over S, where we think of $R^{1/q}[S] \in S^{1/q}$. Let c be an element of R such that $cS^{1/q} \subseteq R^{1/q}[S]$ for all $q = p^n$. Elements of $\mathcal{J}_{S/R}$ will have this property. Then for every ideal I of S, $cI^* \subseteq I$. Hence, every nonzero element of $\mathcal{J}_{S/R}$ is a test element for tight closure.*

Proof. Since $R^{1/q}$ is faithfully flat over R, $B = R^{1/q} \otimes_R S$ is faithfully flat over S. Hence, elements of $S - \{\{0\}$ and of $R - \{0\}$ are nonzerodivisors on B, and if tensor with $\mathcal{K} = (R)$ we obtain that $R^{1/q} \otimes_R S \subseteq \mathcal{K} \otimes_R (R^{1/q} \otimes_R S) \cong (\mathcal{K} \otimes_R \mathcal{K}) \otimes_R (R^{1/q} \otimes_R S) \cong$ (by the associativity of tensor) $(\mathcal{K} \otimes_R R^{1/q}) \otimes_R (\mathcal{K} \otimes_R S) \cong \mathcal{K}^{1/q} \otimes_R \mathcal{L}$, where $\mathcal{L} = (L)$. (With an integral extension domain D of a domain R, inverting every element of $R - \{0\}$ inverts every element of $D - \{0\}$, since each nonzero element of D has a nonzero multiple in R.) The tensor product of two modules over $W^{-1}R$ is the same whether the base is taken to be R or $W^{-1}R$, so the last term becomes $\mathcal{K}^{1/q} \otimes_{\mathcal{K}} \mathcal{L}$. Because L is separable over $\mathcal{K}$, $\mathcal{L} \cong \mathcal{K}[X]/(F)$ where F is a separable monic polynomial in X, and the tensor product is $\mathcal{K}^{1/q}[X]/(F)$, which is reduced. Thus, $\mathcal{K}^{1/q} \otimes L$ is reduced. Every element u has the form $\sum_{j=1}^{t} \alpha_j^{1/q} \otimes \lambda_j$ with the $\alpha_j \in \mathcal{K}$ and the $\lambda_j \in \mathcal{L}$. Then $u^q \sum_{j=1} t\alpha \otimes \lambda^q = 1 \otimes \sum_{j=1}^{t} \alpha_j \otimes \lambda^q \in \mathcal{L}$, and so is a unit if it is not zero. Since $\mathcal{K}^{1/q} \otimes_{\mathcal{K}} \mathcal{L}$ is reduced, every nonzero element u has a nonzero power in $\mathcal{L}$ that is a unit, and so every nonzero element is a unit. Thus, $\mathcal{K}^{1/q} \otimes_{\mathcal{K}} \mathcal{L}$ is a field, and the map $\mathcal{K}^{1/q} \otimes_{\mathcal{K}} \mathcal{L} \to \mathcal{L}^{1/q}$ induced by $\mathcal{K}^{1/q} \subseteq \mathcal{L}^{1/q}$ and $\mathcal{L} \subseteq \mathcal{L}^{1/q}$ is injective, with image $\mathcal{F} = \mathcal{K}^{1/q}[L]$. Thus, $R^{1/q} \otimes_R S \to S^{1/q}$ is injective, with image $B = R^{1/q}[S]$. We claim that $\mathcal{F} = \mathcal{L}^{1/q}$. To see this note that $[\mathcal{F} : \mathcal{K}^{1/q}] = [L : K] = [L^{1/q} : K^{1/q}]$. Hence, the fraction field of $R^[1/q][S]$ is the same as the fraction field of $S^{1/q}$, namely, $\mathcal{L}^{1/q}$. Hence, $S^{1/q}$ is contained in the integral closure of $R^{1/q}[S]$. By the Lipman-Sathaye theorem, $\mathcal{J}_{B/R^{1/q}} S^{1^q} \in B$. But if $S = T/(F_1, \ldots, F_m)$ is a presentation over R, where T is a polynomial ring over R, then $B = R^{1/q} \otimes T/(F_1, \ldots, F_m)$ is a presentation over $R^{1/q}$. It follows that $\mathcal{J}_{B/R^{1/q}} = \mathcal{J}_{S/R} B$, and so $\mathcal{J}_{S/R} S^{1/q} \subseteq B$.

Now suppose that c is any element in S such that $cS^{1/q} \in R^{1/q}[S]$ for all $q = p^n$. If $s \in I^*$, there exists $d \in S - \{0\}$ such that $ds^q \in I^{[q]}$ for all $q \gg 0$. Every nonzero element of S has a multiple in R. Hence, *we may replace d by a nonzero multiple in R and henceforward assume that $d \in R - \{0\}$*. Taking qth roots yields $d^{1/q}s \in IS^{1/q}$ and hence $cd^{1/q} in I(cS^{1/q}) \subseteq IB$, where $B = R^{1/q}[S]$. Then $d^{1/q} \in IB :_B cs = IB :_B csB = (I :_S cs)B$ since $d^{1/q} \in B$ and B is S-flat, and this holds for all $q \gg 0$. But then $d \in (I :_S cs)^{[q]}$ for all $q \gg 0$. If $I :_S cs$ is a proper ideal of S, this is impossible: if we local at maximal ideal containing $I :_S cs$, $d \neq 0$ will be in every power of the maximal ideal of the local ring obtained. Hence, $I :_S cq = S$, i.e., $cs \in I$. □