Math 615, Winter 2015 **Problem Set #1**
Due: Friday, January 30

1. Let $S = K[x, y]$ be a polynomial ring over the field K with the usual grading and let $R = K[x^2, xy, y^3] \subseteq S$ with the $\mathbb{N}$-grading inherited from S. Find the Poincaré series $\sum_{n=0}^{\infty} \dim_K([R_n])t^n$ as a rational function of t. Is the graded Hilbert function of R, $H(n) = \dim_K([R]_n)$, eventually polynomial?

2. Let R be a ring and M an R-module. Then $x \in R$ is a *zerodivisor on M* if there is an element $u \in M - \{0\}$ such that $xu = 0$. $x_1, \ldots, x_n$ is called a *possibly improper regular sequence* on the R-module M if x_{i+1} is not a zerodivisor on $M/(x_1, \ldots, x_i)M$ for $0 \leq i \leq n-1$. That is, x_1 is not a zerodivisor on M, x_2 is not a zerodivisor on M/x_1M, and so forth. A possibly improper regular sequence on M is called a *regular sequence* on M if, in addition, $(x_1, \ldots, x_n)M \neq M$. An important case occurs when $M = R$. Show that if $x_1, \ldots, x_n$ is a possibly improper regular sequence on M, then x_i is not a zerodivor on $M/\mathfrak{A}_iM$, where $\mathfrak{A}_i$ is the ideal generated by all of the x_j for $j \neq i$. (Regular sequences may lose that property when permuted: $z, x(1-z), y(1-z)$ is a regular sequence in the polynomial ring $K[x, y, z]$, but $x(1-z), y(1-z), z$ is not.)

3. Let I be an ideal of a ring R such that $\cap_{n=1}^{\infty} I^n = (0)$. Show that if the associated graded ring $\mathrm{gr}_I R$ is a domain, then R is a domain.

4. If R is Noetherian graded by $\mathbb{N}^n$ or $\mathbb{Z}$ and M is a $\mathbb{Z}^n$-graded module, then by a class theorem the associated primes of M are graded. Is this true for $\mathbb{Z}/2\mathbb{Z}$-gradings? (Prove that it is true, or give a counter-example.) Note that one will have $R = R_0 \oplus R_1$ where R_0 is a subring, R_1 is an R_0-module, and the product of two elements of R_1 is in R_0.

5. Let t, x, y, z be formal power series indeterminates over the field K, let $T = K[[x, y, z]]$, and let $P = (x^2 - y^3, x^3 - z^5)T$. Show that P is the kernel of the map $T \to K[[t]]$ sending $f(x, y, z) \to f(t^{15}, t^{10}, t^9)$. The image of this map is $R = K[[t^{15}, t^{10}, t^9]] \subseteq K[[t]]$. Give generators and relations over K for $\mathrm{gr}_\mu(R)$, where μ is the maximal ideal of R.

6. Let R, S be finitely generated $\mathbb{N}$-graded A-algebras with $R_0 = S_0 = A$. Let $R^{(h)} = \oplus_{n=0}^{\infty}[R]_{hn}$ $\mathbb{N}$-graded so that $[R^{(h)}]_n = [R]_{hd}$ (the hth *Veronese subring* of R). Then $T = R \otimes_A S$ is $\mathbb{N} \times \mathbb{N}$-graded such that $[T]_{(n,n')} = [R]_n \otimes_A [R]_{n'}$ and $\mathbb{N}$-graded with $[T]_n = \bigoplus_{i+j=n}[T]_{(i,j)}$. The *Segre product* $R \circledS_A S := \bigoplus_{n=0}^{\infty}[T]_{(n,n)}$ is $\mathbb{N}$-graded with $[R \circledS_A S]_n = [T]_{(n,n)}$. If R, S are standard, so are $R^{(d)}$, T with its $\mathbb{N}$-grading, and $R \circledS_A S$.

If $A = K$ is a field and R, S are standard with Krull dimensions d, d' and multiplicities e, e', respectively (in the graded case, this means the multiplicity of the local ring at the homogeneous maximal ideal), are the multiplicities of $R^{(h)}$, T, and $R \circledS_K S$ determined? Prove your answer, and give formulas for those which are determined.

Extra Credit 1. If $F = x^2 - y^2 + x^3$ and $G = x^3 + y^3 + y^4$, find the intersection multiplicity of the curves $V(F)$ and $V(G)$ at the origin in $\mathbb{A}^2_K$ over the algebraically closed field K. That is, with $m = (x, y)$ and $R = K[x, y]_m$, determined the length (or K-vector space dimension) of $R/(F, G)$.

Extra Credit 2. With notation as in the first paragraph of Problem **6.**, prove that there exists a choice of d such that $R^{(d)}$ is standard.