Math 615, Winter 2015 **Problem Set #1: Solutions**

1. Consider the K-algebra map $K[U,V,W] \twoheadrightarrow K[x^2, xy, y^3]$ such that U, V, W are sent to x^2, x^3, xy, respectively. The irreducible polynomial $V^6 - U^3W^2$ is in the kernel, which must be a height one prime, since the target ring is a domain of dimension 2. The map is degree preserving if U, V, W have degrees 2, 2, 3, respectively. By a class result, the Poincaré series of the polynomial ring is $1/(1-t^2)(1-t^2)(1-t^3)$. Since the generator of the ideal has degree 12, by a class result the Poincaré series of the quotient is $(1-t^{12})/(1-t^2)^2(1-t^3) = (1-t^3)(1+t^3)(1+t^6)/(1-t)^2(1+t)^2(1-t^3) = (1+t)(1-t+t^2)(1+t^6)/(1-t)^3(1+t)^2 = (1-t+t^2)(1+t^6)/(1-t)^3(1+t)$. For degree $n = 6k$, $k \geq 1$, one has all monomials in x and y of degree n, except xy^{n-2}, xy^{n-1}: one may check $k = 1$ easily and then use induction on k. Thus, the Hilbert function for $6|n$ is $n+1-2 = n-1$, and if the Hilbert function were eventually polynomial, it would agree with $n-1$ for all $n \gg 0$. But this is false for all $n = 6k+5$, since x^n, xy^{n-1}, and y^n are all missing from the ring.

2. Use induction on n. The case $n = 1$ is clear: if $x_1m_1 = 0$, then $m_1 = 0$, while $\mathfrak{A}_1 = (0)$. It suffices to show that if $n \geq 2$ and $(*)$ $\quad\sum_{j=1}^{n} x_jm_j = 0$, then $m_i \in \mathfrak{A}_iM$. If $i = n$ this follows at once from the definition of a possibly improper regular sequence. If $i < n$, then we have $m_n = \sum_{j=1}^{n-1} x_ju_j$, and we may rewrite the given relation as $\sum_{j=1}^{n-1} x_j(m_j + x_nu_j) = 0$. By the induction hypothesis, $m_i + x_nu_i \in \mathfrak{B}_iM$, where $\mathfrak{B}_i$ is generated by the elements x_j for $j \neq i$ and $1 \leq j \leq n-1$. Since $x_n \in \mathfrak{A}_i \supseteq \mathfrak{B}_i$, $m_i \in \mathfrak{A}_iM$. $\square$

3. If $r, s \neq 0$ in R and $rs = 0$, then, since $\bigcap_{n=1}^{\infty} I^n = (0)$, we can choose integers $a, b \in \mathbb{N}$ such that $r \in I^a - I^{a+1}$ and $s \in I^b - I^{b+1}$ (here, $I^0 := R$). Then $[r] \in I^a/I^{a+1}$ and $[s] \in I^b/I^{b+1}$ are nonzero elements of $\mathrm{gr}_I(R)$ whose product is 0, a contradiction. $\square$

4. Let K be any field and let $M = R = K[X]/(X^2-1) \cong K + Kx$, where X is an indeterminate over K and x is its image. This ring is $\mathbb{Z}/2\mathbb{Z}$-graded if we let $R_0 = K$ and $R_1 = Kx$, where $\mathbb{Z}_2 = \{0, 1\}$. Then $(x-1)$ is an associated prime (in fact, a minimal prime) of 0 that is not homogeneous.

5. First consider the problem for the map of polynomial rings $T_0 = K[x,y,z] \to K[t]$ obtained by restriction. We first study the K-algebra map of polynomial rings $K[x,z] \to K[t]$ obtained by sending $x \mapsto t^{15}$, $z \mapsto t^9$. The image is a one-dimensional domain, so the kernel must be a height one prime, and contains $x^3 - z^5$. Since the latter is irreducible, we have that $A = K[x,z]/(x^3 - z^5) \cong K[t^{15}, t^9] = B$. We next observe that $y^3 - (t^{15})^2$ is prime in $B[y]$. To see this, it suffices to sow that this is the contraction of the prime $y - t^{10}$ from $K[t][y]$. Suppose $g(y) \in B[y]$ is in the contraction. We can use the division algorithm to divide by $y^3 - (t^{15})^2$ over $B[y]$. The remainder will be of degree at most 2 in y, say $by^2 + cy + d$ where $b, c, d \in B$ and vanish for $y = t^{10}$, i.e., $bt^{20} + ct^{10} + d = 0$. We want to show that $b = c = d = 0$. If not, there is a nonzero relation on t^{20}, t^{10}, and 1 over B, and by taking homogeneous components we get a homogeneous relation in which each coefficient is 0 or t^h with of the form $15r + 9s$. It $d = 0$, we have that $t^{10} = -c/b$. Otherwise, at least one of the terms bt^{20}, ct^{10} has the same degree as d, and we find that t^{10} or t^{20} is the ratio of two powers of t in B. This is not possible, since all powers of t in B are divisible by 3. It follows the $K[x,y,z]/P_0$, where $P_0 = (x^2 - y^3, x^3 - z^5)K[x,y,z]$ is a domain that maps onto $R_0 = K[t^{15}, t^{10}, t^9]$, and both have dimension 1. Hence, they

are equal. Now one may complete with respect to the respective homogeneous maximal ideals to get the required isomorphism in the complete case.
If $\mathcal{M}$ is maximal in the ring S, $S_{\mathcal{M}}/(\mathcal{M})S^n_{\mathcal{M}}]congS/\mathfrak{m}^n$, since the latter has a unique maximal ideal and so elements of $S-\mathcal{M}$ are already invertible. It follows that $\mathrm{gr}_{\mathcal{M}}S_{\mathcal{M}}(S_{\mathcal{M}}) \cong \mathrm{gr}_{\mathcal{M}}S$ are isomorphic as graded rings. Hence, it suffices to study $\mathrm{gr}_{\mathfrak{m}_0}R_0$ for $R_0 = K[t^{15}, t^{10}, t^9]$ with $m_0 = (t^{15}, t^{10}.t^9)$, which is $\cong B/vB$ where $B = K[t^{15}u, t^{10}u, t^9u, v] \subseteq K[t, u, 1/u]$ and $v = 1/u$, using the class fact that $\mathrm{gr}_I(A) \cong A[Iu, v]/(v)$. Thus, we seek generators for the relations on the four monomials $t^{15}u, t^{10}u, t^9u, v = 1/u$, i.e., generators for the kernel Q of the map $K[X, Y, Z, V] \twoheadrightarrow B$. This prime Q contains $X^2 - VY^3, X^3 - V^2Z^5, Y^9 - VZ^{10}$, and $XY^3 - VZ^5$. Any other relation can be rewritten mod these as one that is linear in X or does not involve X and if X occurs the exponent on Y in that term is at most 2. In any relation, the sum of the coefficients of the monomials that map a to given monomial in $K[t, u, 1/u]$ must be 0. Hence, every relation is a linear combination of relations that are the difference of two monomials in X, Y, Z, V that map to the same monomial in $K[t, u, u^{-1}]$. Since Q is a prime not containing any of X, Y, Z, V, in looking for generators we may assume that we have the difference of two monomials with no common factor. We cannot have one of the two monomials be V^s, since the other would map to an element of nonnegative degree in u. If X, Y occur in one monomial, the other must involve Z. Since we may assume that the exponent on X is 1 and the exponent on Y is 1 or 2, the degree in t is $15 + 10$ or $15 + 20$, which is impossible, since the degree in t of any power of Z is a multiple of 9. Therefore, if X occurs, it is with a power of Z, and we must have a power of Y in the other term. If the exponents on Y and Z are a and b, we have $10a = 9b + 15$ so that $b = 5k$ with k odd. Then $b = 10h + 5$, $h \geq 0$. and $a = 9k + 6$. If $h = 0$ we have $VY^6 - Z^5X$. If $h = 1$ we have $Y^{15} - Z^{15}X$. If $h \geq 2$ we have $Y^{9k+6} - V^{h-1}Z^{10k+5}$, which can be obtained from $Y^{15} \equiv Z^{15}X$ by multiplying by the result of raising both sides of $Y^9 = VZ^{10}$ to the $h-1$ power and multiplying. Finally, if X does not occur, Y and Z must occur in the two terms with exponents $9k$ and $10k$ to make the degrees in t match, and that forces V to occur with exponent k, so that what we get is $Y^{9k} - V^kZ^{10k}$, $k \geq 1$. All of these relations are multiples of $Y^9 - VZ^{10}$. We may now kill V in the generators for Q to see that $\mathrm{gr}_{\mu}(R) \cong K[X, Y, Z]/(X^2, Y^9, XY^3, XZ^5)$.

6. Let F, G be the graded Hilbert polynomials of R, S, respectively, whose respective degrees are $d-1$, $d'-1$, and whose respective leading coefficients are $e/(d-1)!$ and $e'/(d'-1)!$ if $d, d' \geq 1$. The Hilbert polynomial of $R \circledS_K S$ is clearly FG, whose degree $d + d' - 2$ (which shows that the Krull dimension is $d + d' - 1$) and whose leading coefficient is $c = ee'/(d-1)!(d'-1)!$. Hence, the multiplicity of the Segre product is $(d + d' - 2)!c = \binom{d+d'-2}{d-1}ee'$. The Hilbert polynomial of $R^{(h)}$ is clearly $F(hn)$, whose leading coefficient is $h^{d-1}/(d-1)!$, and so the multiplicity is h^{d-1}. If R or S has dimension 0, so does the Segre product. The multiplicity of $R \circledS_K S$ in the case of dimension 0 is $\dim_K(R \circledS_K S)$ and is not determined by the specified data; neither is the multiplicity of $R^{(h)}$ if $\dim(R) = 0$.
To study T, if P, Q are two integer valued functions on $\mathbb{N}$, let $P * Q$ denote denote the function whose value on n is $\sum_{i=0}^n P(i)Q(n-i)$. This is a new integer valued function on $\mathbb{N}$ and the operation $*$ is bilinear. If P and Q are the graded Hilbert functions of R, S, respectively, it is immediate that $P * Q$ is the graded Hilbert function of $R \otimes_K S$. Note that $P = F + \delta$ and $Q = G + \epsilon$, where δ, ϵ are functions that are eventually

zero, and $P * Q = F * G + F * \epsilon + \delta * G + \delta * \epsilon$. Note that because $K[x_1, \ldots, x_d] \otimes_K K[x_1, \ldots, x_{d'}] \cong K[x_1, \ldots, x_{d+d'}]$, we have that if $C_d(n) = \binom{n+d-1}{d-1}$ and $C_{d'}(n) = \binom{n+d'-1}{d'-1}$ then $C_d * C - \_d' = C_{d+d'}$. The functions $C_d(n)$ are a $\mathbb{Q}$-basis for the polynomials in n over $\mathbb{Q}$ as d varies in $\mathbb{N}$, since C_d has is the unique element of this family of degree d. Since F, G have the same highest degree terms as eC_d and $e'C_{d'}$, respectively, $F * G$ has highest degree term $ee'C_{d+d'}$. Note that $F * \epsilon$ is a sum of polynomials of the same degree as F whose leading coefficient is the sum of the values of ϵ multiplied by the leading coefficient of F, while $\delta * \epsilon$ is eventually 0. It follows that if neither dimension is 0, the multiplicity of T is ee'. This is true in all cases. If $d \neq 0$ while $d' = 0$, the coefficients of $G = \epsilon$ are all positive, and their sum if the multiplicity of S, and the argument for $d = 0$, $d' \neq 0$ is similar. The check is likewise straightforward if $d = d' = 0$.

EC1. The intersection multiplicity e is 7 if $\mathrm{char}(K) \neq 2$ and 6 if the $\mathrm{char}(K) = 2$. Completing the ring does not affect the length. Assume $\mathrm{char}(K) \neq 2$. Then $K[[x, y]]$ contains a square roots $\pm u \in K[[x]]$ for $1+x$ where $u = 1+\frac{1}{2}x+\cdots$. $-F = (y-xu)(y+xu)$, and $y - xu$, $y + xu$ are prime in $K[[x, y]]$ (both quotients are $K[[x]]$). The images g, h of G are $x^3 \pm (xu)^3 + (xu)^4 = x^3(1 \pm u^3 + xu^4)$. If g corresponds to $+$, its lowest degree term is $2x^3$ and $g = x^3v$ where $v \in K[[x]]$ is unit. $h = x^3(1 - (1 + 3/2x + \cdots) + x(1 + \cdots)) = x^3(-\frac{1}{2}x + \cdots)$ which is x^4w where $w \in K[[x]]$ is a unit. Hence, each of $y - xu$, G and $y + xu, G$ is a regular sequence, and, by Problem #2. above, $a = y - xu$ and $b = y + xu$ are nonzerodivisors in $A = K[[x, y]]/G$. In general, if a, b are nonzerodivisors in A, then $0 \to aA/abA \to A/abA \to A/aA \to 0$ is exact, and since $aA/abA \cong A/bA$, we have that $(\dagger)$ $\ell(A/abA) = \ell(A/aA)+\ell(A/bA)$. Hence, $\ell\big(K[[x.y]]/(F, G)\big) = \ell(A/abA) = \ell(A/aA)+ \ell(A/bA) = \ell(K[[x, y]]/(y - ux, G)) + \ell(K[[x, y]]/(y + ux, G)$. These are $\ell\big(K[[x]]/(g)\big) = \ell\big(K[[x]]/(x^3)\big) = 3$ and $\ell\big(K[[x]]/(h)\big) = \ell\big(K[[x]]/(x^4)\big) = 4$. Hence, $e = 3 + 4 = 7$.
If $\mathrm{char}(K) = 2$, let $z = x + y$. Then $x = z + y$. The ring is $K[[z, y]]$, $F = z^2 + y^3$, and $G = (y+z)^3 + y^3 + y^4 = y^3 + y^2z + yz^2 + z^3 + y^3 + y^4 = y^2z + yz^2 + y^3 + y^4 = y^2z + y^3 + yF$, so that $(F, G) = (z^2 + y^3, y^2z + y^3)$. Then $y^2z + y^3 = yy(y+z)$. The the elements $y, y, y+z$ are prime, all with quotient $K[[z]]$, and the respective images of $z^2 + y^3$ are z^2, z^2, and $z^2(1 + z)$. The iterated use of $(\dagger)$ from the previous paragraph yields that the length of $K[[x, y]]/(F, G) = K[[z, y]]/\big(yy(y + z), z^2 + y^3\big)$ is the sum of the three lengths one obtains when kills z^2, z^2, $z^2(2 + z)$ respectively in $K[[z]]$. Thus, $e = 2 + 2 + 2 = 6$.

EC2. Let the degrees of the generators be $a_1, \ldots, a_s$, let L be a common multiple of $a_1, \ldots, a_s$ and let $L_i = L/a_i$. Let $h = sL$. We show $R^{(h)}$ is standard. by proving that all monomials in the generators of degree kh, $k \geq 1$, are products of such monomials of degree h. If the exponents on the generators are $m_1, \ldots, m_s$, then $(*)$ $a_1b_1 + \cdots + a_sb_s = kh$. By induction on k, it suffices to show that if $k > 1$ we can find $c_1, \ldots, c_s \in \mathbb{N}$ such that $0 \leq c_i \leq b_i$ and $a_1c_1+\cdots+a_sc_s = h$: we can then factor the given monomial as one in which the exponents are the c_i and another in which the sum of the exponents is $(k-1)h$. Divide both sides of $(*)$ by L to obtain $\frac{b_1}{L_1} + \cdots + \frac{b_s}{L_s} = ks$. Let $q_i = \lfloor \frac{b_i}{L_i} \rfloor$, so that $\frac{b_i}{L_i} = q_i + f_i$ with $0 \leq f_i < 1$. Then the sum of the f_i is at most s, and so the sum of the $q_i \in \mathbb{N}$ is at least $(k - 1)s \geq s$. We may therefore decrease some of the q_i to nonnegative integers r_i whose sum is s. Take $c_i = r_iL_i \leq q_iL_i \leq b_i$, and $\sum_{i=1}^{s} c_ia_i = \sum_{i=1}^{s} r_iL_ia_i = \sum_i r_iL = sL = h$, as required. $\square$