Math 615, Winter 2014 **Problem Set #2**
Due: Monday, February 23

1. Let M and N be modules over a ring R, let $x \in R$ be in the annihilator of N, and suppose that x is not a zerodivisor on M. Show that for all i, there is an exact sequence

$$0 \to \mathrm{Tor}_i(M, N) \to \mathrm{Tor}_i(M/xM,\, N) \to \mathrm{Tor}_{i-1}(M,\, N) \to 0.$$

Hence, $\mathrm{Tor}_i(M/xM, N) = 0$ if and only if $\mathrm{Tor}_i(M, N)$ and $\mathrm{Tor}_{i-1}(M, N)$ are both 0.

2. Let M and N be modules over a ring R, let $x_1, \ldots, x_d \in R$ be in the annihilator of N, and suppose that $x_1, \ldots, x_d$ is a possibly improper regular sequence on M. Show that $\mathrm{Tor}_i(M, N) = 0$ for $i \geq n$ if and only if $\mathrm{Tor}_i(M/(x_1, \ldots, x_d)M,\, N) = 0$ for $i \geq n + d$.

3. Let $0 \to A \to B \to C \to 0$ be an exact sequence of modules over the ring R, let $f \in R - \{0\}$ and let M be an R-module such that M_f is flat over R, or such that C_f is flat over R. Show that if f is not a zerodivisor on $A \otimes_R M$, then $A \otimes_R M \to B \otimes_R M$ is injective.

4. Let (R, m, K) be the local ring $K[[x, y]]/(x^2, xy)$. Give a recursive description of the minimal modules of syzygies arising in a minimal free resolution $F_\bullet$ of $K = R/m$ over R. Also give a recursive description of the sequence of ranks of the free modules in $F_\bullet$, which are the same as the K-vector space dimensions of the modules $\mathrm{Tor}_n^R(K, K)$.

5. Let R be a local ring and let M be a finitely generated R-module of projective dimension n. Let $R \to S$ be a local homomorphism (so that the maximal ideal of R maps into the maximal ideal of S). Suppose that $\mathrm{Tor}_i(M,\, S) = 0$ for $i \geq 1$. (This holds if S is R-flat or if $S = R/(x_1, \ldots, x_d)$ where $x_1, \ldots, x_d$ is a regular sequence on both R and M.) Show that the projective dimension of $S \otimes_R M$ over S is also n.

6. Let $0 \to A \to B \to C \to 0$ be a sequence of nonzero finitely generated modules over a local ring R. Use the long exact sequence for $\mathrm{Tor}_\bullet^R(\_,\, K)$ to show that:
(a) $\mathrm{pd}_R(B)$ is at most the greater of $\mathrm{pd}_R A$ and $\mathrm{pd}_R C$, and that
(b) if $\mathrm{pd}_R B < \mathrm{pd}_R C$, then $\mathrm{pd}_R A = \mathrm{pd}_R C - 1$.

Extra Credit 3. Let (T, m) be a local ring of Krull dimension $d + 1$ and let f be a nonzerodivisor in m. Let $R = T/fR$. Let M be a finitely generated nonzero R-module that has finite projective dimension over T (always true if T is regular). Show that a dth module of syzgies Q of M over R is 0 or else has projective dimension 1 over T, with Betti numbers equal.

Extra Credit 4. With T, f, R and Q as in Extra Credit Problem #3 just above, let $0 \to T^n \xrightarrow{\alpha} T^n \to Q \to 0$ be a minimal free resolution of Q over T, where α is a size n square matrix over T. Let $e_1, \ldots, e_n$ be the standard basis for T_n. Then fe_j is in the column space of the matrix α for every j (explain why), and can be written αv_j for some $n \times 1$ column v_j. Let I_n be the size n identity matrix over T. Show that $fI_n = \alpha\beta$, where the columns of β are the v_j. Explain why $\beta\alpha = fI_n$ as well. Prove that

$$\cdots \xrightarrow{\beta} R^n \xrightarrow{\alpha} R^n \xrightarrow{\beta} R^n \xrightarrow{\alpha} R^n \to Q \to 0$$

gives a free resolution of Q over R.