Math 615, Winter 2014 **Problem Set #3**
Due: Monday, March 30

1. Let R be a ring and M an R-module. If $\underline{x}^- = x_1, \ldots, x_{n-1} \in R$, $y \in R$, and $z \in R$, show there is a long exact sequence

$$\cdots \to H_i(\underline{x}^-,\, y; M) \to H_i(\underline{x}^-,\, yz; M) \to H_i(\underline{x}^-,\, z; M) \to \cdots.$$

Deduce that if R is local, M is finitely generated, and all three are defined, then $\chi(\underline{x}^-,\, yz; M) = \chi(\underline{x}^-,\, y; M) + \chi(\underline{x}^-,\, z; M)$. (Spectral sequences are not needed for this.)

2. Consider a map ϕ of complexes $B_\bullet \to A_\bullet$, so that $\phi_p : B_p \to A_p$, and let $C_\bullet$ be the total complex of the double complex D_{pq} whose 0 th row is $A_\bullet$, i.e., $D_{p0} = A_p$, whose first row is $B_\bullet$, i.e, $D_{p1} = B_p$, and who other rows are 0, where the map $B_p \to A_p$ is $(-1)^p\phi_p$. Consider the spectral sequence obtained by filtering D_{pq} by columns. Show that E^2_{p0} is the homology of the row of cokernels $\mathrm{Coker}\,(\phi_\bullet)$, and E^2_{p1} is the homology of the row of kernels, $\mathrm{Ker}\,(\phi_\bullet)$, while all other E^2_{pq} are 0. Show that there is a long exact sequence

$$\cdots \to H_{n-1}\big(\mathrm{Coker}\,(\phi_\bullet)\big) \to H_n(C_\bullet) \to H_n\big(\mathrm{Ker}\,(\phi_\bullet)\big) \to H_{n-2}\big(\mathrm{Coker}\,(\phi_\bullet)\big) \to \cdots$$

3. Let $A_\bullet$ be a finite complex $0 \to A_n \to \cdots \to A_0 \to 0$ of flat R-modules over the ring R such that $H_j(A_\bullet)$ is annihilated by $I_j \subseteq R$, $0 \le j \le n$. Let M be an arbitrary R-module. Show that the product ideal $\prod_{j=0}^s I_j$ annihilates $H_s(A_\bullet \otimes M)$.

4. Let $K[s,t]$ be polynomial over the field K, and let R be the local ring of $K[s^4, s^3t, st^3, t^4] \subseteq K[s,t]$ at the graded maximal ideal. Determine the Koszul homology modules of R with respect to system of parameters $x = s^4$, $y = t^4$, including their lengths. Determine $\chi(x, y;\, R)$. Is R Cohen-Macaulay?

5. Let R be a finitely generated standard $\mathbb{N}$-graded algebra over a field K (so that $R_0 = K$ and R is generated by its 1-forms). Let m be the homogeneous maximal ideal in R. Assume that $x_1, \ldots, x_d \in [R]_1$ is a homogeneous system of parameters, where $d \ge 2$. Let $S = R \,ⓢ_K\, K[y,z]$ be the Segre product of R with a polynomial ring in two variables over K. Note that $\dim(S)$ is $\dim(R)+1$ by the solution to Problem **6.** of the first problem set.

(a) Show that S is isomorphic with the Rees ring $R[mt]$ of R with respect m.

(b) Show that $x_1z, x_1y - x_2z, \ldots, x_iy - x_{i+1}z, \ldots, x_{d-1}y - x_dz, x_dy$ is a homogeneous system of parameters for S.

6. Let (A, m) be an Artin local ring, and $x_1, x_2 \in m$. Does $H_1(x_1,\, x_2;\, A)$ need at least two generators? Prove your answer.

Extra Credit 5. Keep the notation of Problem **5.** above. Show that if $R = K[x_1, \ldots, x_d]$ is polynomial, then S is Cohen-Macaulay. Also show that if $R = K[X_1, \ldots, X_n]/(F)$, where $X_1, \ldots, X_n$ are indeterminates, F is monic as a polynomial in X_n of degree r, so that the images of $X_1, \ldots, X_{n-1}$ form a homogeneous system of parameters, and $r \ge n$, then S is not Cohen-Macaulay.

Extra Credit 6. Let $x_1, \ldots, x_s$ be a regular sequence in R and $I = (x_1, \ldots, x_s)$. Let $\mathcal{F}_\bullet$ denote a left flat complex all of whose homology is killed by I. Show that for all i, there is a surjection $H_{i+s}(R/I \otimes_R \mathcal{F}_\bullet) \twoheadrightarrow H_i(\mathcal{F}_\bullet)$.