Math 615, Winter 2014 **Problem Set #4**
Due: Monday, April 6

1. Let x, y, X_1, X_2, X_3, X_4 be indeterminates over the field K. The kernel P of the K-algebra map $S = K[X_1, X_2, X_3, X_4] \twoheadrightarrow K[x^2, x^3, xy, y] \subseteq K[x, y]$ such that X_1, X_2, X_3, X_4 are sent to x^2, x^3, xy, y, resp., is the ideal generated by the 2×2 minors of the matrix $\begin{pmatrix} X_1 & X_2 & X_3 & X_4 \\ X_2 & X_1^2 & X_1X_4 & X_3 \end{pmatrix}$, which is $(X_1^3 - X_2^2, X_1^2X_4 - X_2X_3, X_1X_3 - X_2X_4, X_3^2 - X_1X_4^2)$. You may assume this. Let $X = V(P)$ and $Y = V(Q)$, where $Q = (X_2, X_4)$. Show that the intersection $X \cap Y$, as a set, consists only of one point, the origin, corresponding to $m = (X_1, X_2, X_3, X_4)$, and find the intersection multiplicity of X and Y at this point. I.e., with $T = S_m$, $\chi^T(T/PT,\, T/QT)$. What is $\ell\big((T/PT) \otimes T(T/QT)\big)$?

2. Let R be a regular local ring whose completion is formal power series over a field or a discrete valuation ring. Let P, Q be prime ideals such $\dim(R/P) + \dim(R/Q) = \dim(R)$ and $P + Q$ is m-primary. Let M, N be nonzero torsion free modules over R/P and R/Q, respectively, and suppose that the torsion-free rank of M (resp., N) over R/P (resp., over R/Q) is r (resp., s). Show that $\chi^R(M, N) = rs\,\chi^R(R/P, R/Q)$.

3. Let R be a regular local ring and let M, N be nonzero finitely generated modules such that $\ell(M \otimes_R N)$ is finite. Suppose that $\mathrm{depth}_m M = r = \dim(M)$, that $\mathrm{depth}_m N = s = \dim(N)$ (i.e., M and N are Cohen-Macaulay modules), and that $r + s = d = \dim(R)$. Show that $\mathrm{Tor}_i^R(M, N) = 0$ for $i \geq 1$. (Hence, $\chi(M, N) = \ell(M \otimes_R N) > 0$.)

4. Let $(R, m) \hookrightarrow S$ be flat local homomorphism of regular rings, let $\underline{x} = x_1, \ldots, x_d$ be a system of parameters for R, and let $x_1, \ldots, x_d, y_1, \ldots y_r$ be a system of parameters for S. Let $\underline{y} = y_1, \ldots, y_r$, and let $\overline{S} = S/(\underline{y})S$. Show that $\overline{S}$ is flat over R.

5. With (R, m), S, T, $\underline{x}$, and $\underline{y}$ as in problem **4.**, let M, N be nonzero R-modules such that $\ell(M \otimes_R N)$ is finite. Let $B = S \otimes_R M$ and $C = \overline{S} \otimes_R N$. Show that $\chi^S(B, C)$ is defined, and that it is equal to $\ell_S\big(S/(m, \underline{y})\big)\chi^R(M, N)$.

6. Let $R \to S$ be a homomorphism of Noetherian rings such that S has projective dimension at most d as an R-module. Let M be an R-module and let N be an S-module of projective dimension at most h over S. Show that $\mathrm{Tor}_n^R(M, N)$ vanishes for $n > d + h$, and is the same as $\mathrm{Tor}_h^S\big(\mathrm{Tor}_d^R(M, S), N\big)$ for $n = d + h$.

Extra Credit 7. Assume that the normalization of a complete local two-dimensional domain D is a module-finite extension of D. (It will then be a finitely generated D-module of depth two on the maximal ideal of D.) Prove that if M, N are nonzero modules over an arbitrary four-dimensional regular local ring R such that $\ell(M \otimes_R N)\dim(M) + \dim(N) = 4$, then $\chi^R(M, N) > 0$. As observed in class, you may assume that M, N are prime cyclic modules.

Extra Credit 8. Let notation by as in Problem **1**. Determine $\chi^T(T/P^rT,\, T/Q^sT)$ as a function of the positive integers r and s.