DEPTH, COHEN-MACAULAY RINGS, AND FLATNESS

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By a *quasilocal ring* (R, m, K) we mean a ring with a unique maximal ideal m: in this notation, $K = R/m$. A quasilocal ring is called *local* if it is Noetherian. A homomorphism $h : R \to S$ from a quasilocal ring (R, m, K) to a quasilocal ring (S, m_S, K_S) is called *local* if $h(m) \subseteq m_S$, and then h induces a map of residue fields $K \to K_S$.

If $x_1, \ldots, x_n \in R$ and M is an R-module, the sequence $x_1, \ldots, x_n$ is called a *possibly improper* regular sequence on M if x_1 is not a zerodivisor on M and for all i, $0 \leq i \leq n-1$, x_{i+1} is not a zerodivisor on $M/(x_1, \ldots, x_i)M$. A possibly improper regular sequence is called a *regular sequence* on M if, in addition, $(*)\quad (x_1, \ldots, x_n)M \neq M$. When $(*)$ fails, the regular sequence is called *improper*. When $(*)$ holds we may say that the regular sequence is *proper* for emphasis, but this use of the word "proper" is not necessary.

Note that every sequence of elements is an improper regular sequence on the 0 module, and that a sequence of any length consisting of the element 1 (or units of the ring) is an improper regular sequence on every module.

If $x_1, \ldots, x_n \in m$, the maximal ideal of a local ring (R, m, K), and M is a nonzero finitely generated R-module, then it is automatic that if $x_1, \ldots, x_n$ is a possibly improper regular sequence on M then $x_1, \ldots, x_n$ is a regular sequence on M: we know that $mM \neq M$ by Nakayama's Lemma.

If $x_1, \ldots, x_n \in R$ is a possibly improper regular sequence on M and and S is any flat R-algebra, then the images of $x_1, \ldots, x_n$ in S form a possibly improper regular sequence on $S \otimes_R M$. By a straightforward induction on n, this reduces to the case where $n = 1$, where it follows from the observation that if $0 \to M \to M$ is exact, where the map is given by multiplication by x, this remains true when we apply $S \otimes_R \_$. In particular, this holds when S is a localization of R.

If $x_1, \ldots, x_n$ is a regular sequence on M and S is flat over R, it remains a regular sequence provided that $S \otimes_R \big(M/(x_1, \ldots, x_n)M\big) \neq 0$, which is always the case when S is faithfully flat over R.

Nakayama's Lemma, including the homogeneous case

Recall that in Nakayama's Lemma one has a *finitely generated module* M over a quasilocal ring (R, m, K). The lemma states that if $M = mM$ then $M = 0$. (In fact, if $u_1, \ldots, u_h$ is a set of generators of M with h minimum, the fact that $M = mM$ implies that $M = mu_1 + \cdots mu_h$. In particular, $u_h = f_1u_1 + \cdots + f_hu_h$, and so $(1 - f_h)u_h = f_1u_1 + \cdots + f_{h-1}u_{h-1}$ (or 0 if $h = 1$). Since $1 - f_h$ is a unit, u_h is not needed as a generator, a contradiction unless $h = 0$.)

By applying this result to M/N, one can conclude that if M is finitely generated (or finitely generated over N), and $M = N + mM$, then $M = N$. In particular, elements of M whose images generate M/mM generate M: if N is the module they generate, we have $M = N + mM$. Less familiar is the homogeneous form of the Lemma: it does not need M to be finitely generated, although there can be only finitely many negative graded components (the detailed statement is given below).

First recall that if H is an additive semigroup with 0 and R is an H-graded ring, we also have the notion of an H-graded R-module M: M has a direct sum decomposition

$$M = \bigoplus_{h \in H} M_h$$

as an abelian group such that for all $h, k \in H$, $R_h M_k \subseteq M_{h+k}$. Thus, every M_h is an R_0-module. A submodule N of M is called graded (or homogeneous) if

$$N = \bigoplus_{h \in H} (N \cap M_h).$$

An equivalent statement is that the homogeneous components in M of every element of N are in N, and another is that N is generated by forms of M.

Note that if we have a subsemigroup $H \subseteq H'$, then any H-graded ring or module can be viewed as an H'-graded ring or module by letting the components corresponding to elements of $H' - H$ be zero.

In particular, an $\mathbb{N}$-graded ring is also $\mathbb{Z}$-graded, and it makes sense to consider a $\mathbb{Z}$-graded module over an $\mathbb{N}$-graded ring.

Nakayama's Lemma, homogeneous form. *Let R be an $\mathbb{N}$-graded ring and let M be any $\mathbb{Z}$-graded module such that $M_{-n} = 0$ for all sufficiently large n (i.e., M has only finitely many nonzero negative components). Let I be the ideal of R generated by elements of positive degree. If $M = IM$, then $M = 0$. Hence, if N is a graded submodule such that $M = N + IM$, then $N = M$, and a homogeneous set of generators for M/IM generates M.*

Proof. If $M = IM$ and $u \in M$ is nonzero homogeneous of smallest degree d, then u is a sum of products i_tv_t where each $i_t \in I$ has positive degree, and every v_t is homogeneous, necessarily of degree $\geq d$. Since every term i_tv_t has degree strictly larger than d, this is a contradiction. The final two statements follow exactly as in the case of the usual form of Nakayama's Lemma. $\square$

In general, regular sequences are not permutable: in the polynomial ring $R = K[x, y, z]$ over the field K, $x-1$, xy, xz is a regular sequence but $xy, xz, x-1$ is not. However, if M is a finitely generated nonzero module over a local ring (R, m, K), a regular sequence on M is permutable. This is also true if R is $\mathbb{N}$-graded, M is $\mathbb{Z}$-graded but nonzero in only finitely many negative degrees, and the elements of the regular sequence in R have positive degree. In fact:

Lemma. *Suppose that we have either of the following two situations:*

(1) R is a local ring and M is a finitely generated R-module.

(2) R is an $\mathbb{N}$-graded ring, and M is a $\mathbb{Z}$-graded R-module which is nonzero in only finitely many negative degrees.

Then a regular sequence on M which is in the maximal ideal of R in case (1), and consists of forms of positive degree in case (2), is permutable.

Proof. Note that we get all permutations if we can transpose two consecutive terms of a regular sequence. If we kill the ideal generated by the preceding terms times the module, we come down to the case where we are transposing the first two terms. Since the ideal generated by these two terms does not depend on their order, it suffices to consider the case of regular sequences x, y of length 2. The key point is to prove that y is not a zerodivisor on M. Let $N \subseteq M$ by the annihilator of y. If $u \in N$, $yu = 0 \in xM$ implies that $u \in xM$, so that $u = xv$. Then $y(xv) = 0$, and x is not a zerodivsior on M, so that $yv = 0$, and $v \in N$. This shows that $N = xN$, contradicting the appropriate form of Nakayama's Lemma.

The next part of the argument does not need the local or graded hypothesis: it works quite generally. We need to show that x is a nonzerodivisor on M/yM. Suppose that $xu = yv$. Since y is a nonzerodivisor on xM, we have that $v = xw$, and $xu = yxw$. Thus $x(u - yw) = 0$. Since x is a nonzerodivisor on M, we have that $u = yw$, as required. $\square$

The Krull dimension of a ring R may be characterized as the supremum of lengths of chains of prime ideals of R, where the length of the strictly ascending chain

$$P_0 \subset P_1 \subset \cdots \subset P_n$$

is n. The Krull dimension of the local ring (R, m, K) may also be characterized as the least integer n such that there exists a sequence $x_1, \ldots, x_n \in m$ such that $m = \mathrm{Rad}\big((x_1, \ldots, x_n)R\big)$ (equivalently, such that $\overline{R} = R/(x_1, \ldots, x_n)R$ is a zero-dimensional local ring, which means that $\overline{R}$ is an Artinian local ring).

Such a sequence is called a *system of parameters* for R.

One can always construct a system of parameters for the local ring (R, m, K) as follows. If $\dim(R) = 0$ the system is empty. Otherwise, the maximal ideal cannot be contained in the union of the minimal primes of R. Choose $x_1 \in m$ not in any minimal prime of R. In fact, it suffices to choose x_1 not in any minimal primes P such that $\dim(R/P) = \dim(R)$. Once $x_1, \ldots, x_k$ have been chosen so that $x_1, \ldots, x_k$ is part of a system of parameters (equivalently, such that $\dim\big(R/(x_1, \ldots, x_k)R\big) = \dim(R)-k)$), if $k < \dim(R)$ the minimal primes of $(x_1, \ldots, x_k)R$ cannot cover m. It follows that we can choose x_{k+1} not in any such minimal prime, and then $x_1, \ldots, x_{k+1}$ is part of a system of parameters. By induction, we eventually reach a system of parameters for R. Notices that in choosing x_{k+1}, it actually suffices to avoid only those minimal primes Q of $(x_1, \ldots, x_k)R$ such that $\dim(R/Q) = \dim\big(R/(x_1, \ldots, x_k)R\big)$ (which is $\dim(R) - k$).

A local ring is called *Cohen-Macaulay* if some (equivalently, every) system of parameters is a regular sequence on R. These include regular local rings: if one has a minimal set of generators of the maximal ideal, the quotient by each in turn is again regular and so is a domain, and hence every element is a nonzerodivisor modulo the ideal generated by its predecessors. Moreover, *local complete intersections*, i.e., local rings of the form $R/(f_1, \ldots, f_h)$ where R is regular and $f_1, \ldots, f_h$ is part of a system of parameters for R, are Cohen-Macaulay. It is quite easy to see that if R is Cohen-Macaulay, so is R/I whenever I is generated by a regular sequence.

If R is a Cohen-Macaulay local ring, the localization of R at any prime ideal is Cohen-Macaulay. We define an arbitrary Noetherian ring to be *Cohen-Macaulay* if all of its local rings at maximal ideals (equivalently, at prime ideals) are Cohen-Macaulay. We prove all of this in the sequel.

Regular sequences and depth

We say that $x_1, \ldots, x_N \in R$, any ring, is a *possibly improper regular sequence on* the R-module M if x_1 is not zerodivisor on M and for all i, $1 \leq i \leq n-1$, x_{i+1} is not a zerodivisor on $M/(x_1, \ldots, x_i)M$. This is preserved by flat base change: if S is R-flat, the images of the x_i in S will form a possibly improper regular sequence on $S \otimes_R M$. In particular, we may take S to be a localization of R. If, moreover, $(x_1, \ldots, x_n)M \neq M$ we say that $x_1, \ldots, x_n$ is a regular sequence on M. We make the convention that the empty sequence is a possibly improper regular sequence of length 0 on every R-module M, and that it is a regular sequence if $M \neq 0$. The property of being a regular sequence is preserved by flat base change to S if $S \otimes_R \big(M/(x_1, \ldots, x_n)M\big) \neq 0$, which is always the case if S is faithfully flat over R.

We now focus on the case where R is Noetherian and M is a finitely generated R-module. However, it is convenient to also consider the case where M is a finitely generated R-module over a Noetherian R-algebra S. For simplicity, the reader may want to assume for that $S = R$ on first thinking about the results below.

When I is an ideal of R such that $IM \neq M$, it turns out that there are maximal regular sequences on M contained in I, and that all such maximal regular sequences have the same length, called the *depth* of M on I. We need to prove this. The situation treated below is somewhat more general than the one treated in class, where R and S were assume to be the same.

Theorem. *Let $R \to S$ be a homomorphism of Noetherian rings. Let I be an ideal of R and let M be a finitely generated S-module. Then $IM = M$ if and only if $IS + \mathrm{Ann}_S M$ is the unit ideal, i.e. $V(IS) \cap \mathrm{Supp}\,(M) = \emptyset$.*

If $IM \neq M$, then every element of I is a zerodivisor on M if and only if I is contained in the contraction P of an element Q of $\mathrm{Ass}\,_S(M)$ to R. In this case, the empty sequence is a maximal regular sequence on M in I.

Every regular sequence in I on M can be extended to a maximal regular sequence in I on M, and the maximal such regular sequences all have the same length. Moreover, if a regular sequence $x_1, \ldots, x_d$ in I on M is a maximal such regular sequence, this remains true for the image of the sequence $x_1, \ldots, x_d$ in R_P when we localize at the contraction P of an associated prime Q of $\mathrm{Ass}\,_S M/(x_1, \ldots, x_d)M$ such that $I \subseteq P$, replacing R by R_P, S by S_P (respectively by S_Q, I by IR_P, the x_j by their images $x_j/1$ in $IR_P \subseteq R_P$, and M by M_P (respectively by M_Q). Moreover, the images of the x_j are a maximal regular sequence in PR_P on M_P (respectively, M_Q).

Proof. By a result from Math 614, $\mathrm{Supp}\,(S/IS \otimes_S M) = \mathrm{Supp}\,(S/IS) \cap \mathrm{Supp}\,(M)$, which is the same as $V(IS) \cap V(\mathrm{Ann}_S M) = V(IS + \mathrm{Ann}_S M)$.

The ideal $I \subseteq R$ consists entirely of zerodivisors on M if and only if that is true for its image in S. This is equivalent to the assertion that the image of I is contained in the union of the associated primes of M in S, which is equivalent to the assertion that I is contained in the union of the contractions to R of the associated primes of S. This in turn implies that I is contained in one of these contractions.

If $IM \neq M$ we can extend any regular sequence to a maximal one: the process of extending the sequence with no elements of I must terminate, because if the terms of the regular sequence are $x_1, \ldots, x_n, \ldots$, the sequence of ideals $I_n = (x_1, \ldots, x_n)R$ is ascending and so eventually stable.. But if $I_{n+1} = I_n$, i..e., if $x_{n+1} \in I_n$, then since $I_n \subseteq I$ we have $M/I_n M \neq 0$, while the action of x_{n+1} by multiplication is 0, and so is not injective.

Now suppose that $x_1, \ldots, x_d$ is a maximal regular sequence in I. Then I is contained in the contraction P to R of an associated prime Q of $M/(x_1, \ldots, x_d)M$, or else some element of I is a nonzerodivisor on $M/(x_1, \ldots, x_d)M$ and we can extend the sequence. Note that some element $u \in M(x_1, \ldots, x_d)M$ has annihilator Q, and its annihilator in R is P. Thus, we have an injection $R/P \hookrightarrow M/(x_1, \ldots, x_d)M$ as R-modules sending the image of 1 modulo P to the element $u \in M/(x_1, \ldots, x_d)M$. The situation is preserved when we replace R, S, I, M by R_P, S_P (respectively S_Q), IR_P and M_P (respectively, M_Q) and the x_j by their images in $IR_P \subseteq R_P$. Because PR_P consists entirely of zerodivisors on $\big(M/(x_1, \ldots, x_d)M\big)P \cong M_P/(x_1, \ldots, x_d)M_P$, (respectively, on $\big(M/(x_1, \ldots, x_d)M\big)_Q \cong M_Q/(x_1, \ldots, x_d)M_Q$) we even have that $x_1/1, \ldots, x_d/1$ is a maximal regular sequence in PR_P on M_P (respectively, M_Q).

Finally, suppose that we have two maximal regular sequences $x_1, \ldots, x_d$ and $x'_1, \ldots, x'_{d'}$ in I on M. We may assume without loss of generality that $d \leq d'$. We show by induction on d that $d = d'$. Let P be the contraction of an associated prime Q of $M/(x_1, \ldots, x_d)M$ that contains I to R. We may localize at R at P and S, M at Q and this preserves the situation. Thus, we may assume that without loss of generality that $(R, P) \to (S, Q)$ is local. Note that if $d = 0$, P consists entirely of zerodivisors on M, and the result is immediate. Assume that $d \geq 1$. We first consider the case where $d = 1$. We know that x_1 is a maximal regular sequence on M, and we want to show that the nonzerodivisor x_1' is a maximal regular sequence on M. By hypothesis, there is an element $u \in M$ such that the annihilator of the class of u in M/x_1M is P. Since $x_1' \in P$, we have that $x_1'u \in x_1M$, say $x_1'u = x_1v$. Then $rv \in x_1'M$ iff $rv = x_1'w$ and since x_1 is not a zerodivisor on M, this holds iff for some w, $rx_1v = x_1x_1'w$ iff $rx_1'u = x_1x_1'w$ iff $x_1'(ru - x_1w) = 0$ iff $ru = x_1w$, i.e., iff $r \in P$.

Now suppose that $d > 1$. Since $x_1, \ldots, x_{d-1}$ and $x_1', \ldots, x_{d'-1}'$ are both regular sequences that are not maximal, we have that I is not contained in any of the contractions of the associated primes of $M_1 = M/(x_1, \ldots, x_{d-1})M$, and also not any of the contractions of the associated primes of $M_2 = M/(x_1', \ldots, x_{d'-1}')M$. Hence, I is not contained in the union of all of these, and we can choose $y \in I$ that is a nonzerodivisor on both M_1 and on M_2. Thus, $x_1, \ldots, x_{d-1}y$ is a regular sequence on M, and $x_1', \ldots, x_{d'-1}'$ is a regular sequence on M. Since x_d is a maximal regular sequence on M_1, y is maximal regular sequence on M_1 (this is the case $d = 1$), and so $x_1, \ldots, x_{d-1}, y$ is a maximal regular sequence on M. We have that $x_1', \ldots, x_{d'-1}', y$ is a regular sequence on M (it need no longer be maximal). We can now use the permutability of regular sequences in the local case to conclude that $y, x_1, \ldots, x_{d-1}$ is a maximal regular sequence on M (maximality is also preserved by the permutation, since the quotient $M/(y, x_1, \ldots, x_{d-1})M$ does not depend on the order of the elements), and $y, x_1', \ldots, x_{d'-1}'$ is a regular sequence on M. We may now pass to M/yM, and we obtain from the induction hypothesis that $d' - 1 \leq d - 1$, so that $d' = d$ is forced. $\square$

Let $R \to S$ be a homomorphism of Noetherian rings. Let I be an ideal of R and let M be a finitely generated S-module. If $IM \neq M$ we define the *depth* of M on I to be the length of any maximal regular sequence in M on I. If $IM = M$ we make the convention that the depth of M on I is $+\infty$. We use $\mathrm{depth}_I M$ to denote the depth of M on I.

The *height* of an ideal I of a ring R is the infimum of heights of prime ideals containing I.

Proposition. *Let S be a Noetherian ring, let M be a finitely generated S-module, and let $I = \mathrm{Ann}_S M$. Let J be an ideal of S such that $JM \neq M$. Then:*

(a) $\mathrm{Ass}\,(S/I) \subseteq \mathrm{Ass}\,(M)$.

(b) If (S, Q) is local and $x \in J$ is a nonzerodivisor on M, then $\dim(M/xM) = \dim(M) - 1$ *and* $\mathrm{depth}_J(M/xM) = \mathrm{depth}_J(M) - 1$. *(c) If (S, Q) is local and $x_1, \ldots, x_d$ is a regular sequence in J on M, then* $\dim(M/(x_1, \ldots, x_d)M) = \dim(M) - d$ *and* $\mathrm{depth}_J(M/(x_1, \ldots, x_d)M) = \mathrm{depth}_J(M) - d$.

Proof. (a) Let $u_1, \ldots, u_n$ generate M over S. Then $s \mapsto (su_1, \ldots, su_n)$ has kernel I, and so yields an injection $S/I \hookrightarrow M^{\oplus n}$. This shows that $\mathrm{Ass}\,(S/I) \subseteq \mathrm{Ass}\,(M^{\oplus n}) = \mathrm{Ass}\,(M)$.

(b) The statement about depths is clear. To obtain the statement about dimensions, note that $\dim(M) = \dim(S/I)$. Since x is a nonzerodivisor on M, it is not in any associated prime of M, and hence it is not in associated prime of I. Thus, it is part of a system of parameters for S/I, and so $\dim(S/(I + xS)) = \dim(S/I) - 1 = \dim(M) - 1$. But $I + xS$ has the same radical as the annihilator I_1 of $M/xM \cong S/xS \otimes_M$, since the support of the latter is $V(x) \cap V(I) = V(I + xS)$. Thus, $\dim(S/(I + xS)) = \dim(S/I_1)$, since killing the ideal of nilpotent does not affect the Krull dimension of a ring, and the latter is $\dim(M/xM)$.

(c) follows from (b) by a straightforward induction. □

Corollary. *Let $R \to S$ be a homomorphism of Noetherian rings and let I be an ideal of R. Let M be a finitely generated S-module.*

(a) Then $\mathrm{depth}_I M$ *is the infimum of* $\mathrm{depth}_{IR_P} M_P$ *for primes P of R in the support of M/IM, (such primes must contain I), and also the infimum of* $\mathrm{depth}_{IR_P} M_Q$ *for primes Q of S in the support of M/IM with contraction P to R. It is also the infimum of* $\mathrm{depth}_{PR_P} M_P$ *for primes P of R in the support of M/IM and the infimum of* $\mathrm{depth}_{PR_P} M_Q$ *for primes Q of S in the support of M/IM with contraction P to R.*

(b) $\mathrm{depth}_I M \leq \dim(M) \leq \dim(S)$.

(c) $\mathrm{depth}_I M \leq \dim(R)$.

(d) If J is an ideal of S, the depth of M on J is at most the least height of a minimal prime of J in the support of M. Hence, the depth of S on J is at most the height of J.

Proof. (a) is immediate from the last part of the Theorem above, which also permits us to reduce to the case where $(R, P) \to (S, Q)$ is local.

(b) We can pass to the local case as in part (a) without decreasing the depth of M on I, and the dimensions of M and S can only decrease. The result is then immediate from part (c) of the preceding Proposition.

(c) We may reduce to the case where R and S are local without changing the depth, we may replace R and S by their quotients by the annihilators of M in each, since the dimension of R can only decrease. Let M have generators $u_1, \ldots, u_h$ an injection $R \to M^{\oplus h}$ by $r \mapsto (rm_1, \ldots, rm_h)$. There is no loss of generality in assuming that I is the maximal ideal of R. We use induction on the depth of M. Let x be a nonzerodivisor in I on M. Then x is a nonerodivisor in R. The result now follows from the induction hypothesis, since we may replace R, S, and M by R/xR (which has dimension one smaller than R), S/xS, and M/xM (which has depth one smaller).

(d) When we localize at a minimal prime Q of J, the depth can only increase, and is bounded by $\dim(S_Q)$ provided $M_Q \neq 0$. □

Further properties of regular sequences

In the sequel we shall need to make use of certain standard facts about regular sequences on a module: for convenience, we collect these facts here. Many of the proofs can be made simpler in the case of a regular sequence that is *permutable*, i.e., whose terms form a regular sequence in every order. This hypothesis holds automatically for regular sequences on a finitely generated module over a local ring. However, we shall give complete proofs here for the general case, without assuming permutability. The following fact will be needed repeatedly.

Lemma. *Let R be a ring, M an R-module, and let $x_1, \dots, x_n$ be a possibly improper regular sequence on M. If $u_1, \dots, u_n \in M$ are such that*

$$\sum_{j=1}^{n} x_j u_j = 0,$$

then every $u_j \in (x_1, \dots, x_n)M$.

Proof. We use induction on n. The case where $n = 1$ is obvious. We have from the definition of possibly improper regular sequence that $u_n = \sum_{j=1}^{n-1} x_j v_j$, with $v_1, \dots, v_{n-1} \in M$, and so $\sum_{j=1}^{n-1} x_j(u_j + x_n v_j) = 0$. By the induction hypothesis, every $u_j + x_n v_j \in (x_1, \dots, x_{n-1})M$, from which the desired conclusion follows at once $\square$

Proposition. *Let $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_h = M$ be a finite filtration of M. If $x_1, \dots, x_n$ is a possibly improper regular sequence on every factor M_{k+1}/M_k, $0 \le k \le h-1$, then it is a possibly improper regular sequence on M. If, moreover, it is a regular sequence on M/M_{h-1}, then it is a regular sequence on M.*

Proof. If we know the result in the possibly improper case, the final statement follows, for if $I = (x_1, \dots, x_n)R$ and $IM = M$, then the same hold for every homomorphic image of M, contradicting the hypothesis on M/M_{h-1}.

It remains to prove the result when $x_1, \dots, x_n$ is a possibly improper regular sequence on every factor. The case where $h = 1$ is obvious. We use induction on h. Suppose that $h = 2$, so that we have a short exact sequence

$$0 \to M_1 \to M \to N \to 0$$

and $x_1, \dots, x_n$ is a possibly regular sequence on M_1 and N. Then x_1 is a nonzerodivisor on M, for if $x_1 u = 0$, then x_1 kills the image of u in N. But this shows that the image of u in N must be 0, which means that $u \in M_1$. But x_1 is not a zerodivisor on M_1. It follows that

$$0 \to xM_1 \to xM \to xN \to 0$$

is also exact, since it is isomorphic with the original short exact sequence. Therefore, we have a short exact sequence of quotients

$$0 \to M_1/x_1M_1 \to M/x_1N \to M/x_1N \to 0.$$

We may now apply the induction hypothesis to conclude that $x_2, \ldots, x_n$ is a possibly improper regular sequence on M/x_1M, and hence that $x_1, \ldots, x_n$ is a possibly improper regular sequence on M.

We now carry through the induction on h. Suppose we know the result for filtrations of length $h-1$. We can conclude that $x_1, \ldots, x_n$ is a possibly improper regular sequence on M_{h-1}, and we also have this for M/M_{h-1}. The result for M now follows from the case where $h = 2$. $\square$

Theorem. *Let $x_1, \ldots, x_n \in R$ and let M be an R-module. Let $t_1, \ldots, t_n$ be integers ≥ 1. Then $x_1, \ldots, x_n$ is a regular sequence (respectively, a possibly improper regular sequence) on M iff $x_1^{t_1}, \ldots, x_n^{t_n}$ is a regular sequence on M (respectively, a possibly improper regular sequence on M).*

Proof. If $IM = M$ then $I^kM = M$ for all k. If each of I and J has a power in the other, it follows that $IM = M$ iff $JM = M$. Thus, we will have a proper regular sequence in one case iff we do in the other, once we have established that we have a possibly improper regular sequence. In the sequel we deal with possibly improper regular sequences, but for the rest of this proof we omit the words "possibly improper."

Suppose that $x_1, \ldots, x_n$ is a regular sequence on M. By induction on n, it will suffice to show that $x_1^{t_1}, x_2, \ldots, x_n$ is a regular sequence on M: we may pass to $x_2, \ldots, x_n$ and $M/x_n^{t_1}M$ and then apply the induction hypothesis. It is clear that $x_1^{t_1}$ is a nonzerodivisor when x_1 is. Moreover, $M/x_1^{t_1}M$ has a finite filtration by submodules $x_1^jM/x_1^{t_1}M$ with factors $x_1^jM/x_1^{j+1}M \cong M/x_1M$, $1 \leq j \leq t_1 - 1$. Since $x_2, \ldots, x_n$ is a regular sequence on each factor, it is a regular sequence on $M/x_1^{t_1}M$ by the preceding Proposition.

For the other implication, it will suffice to show that if $x_1, \ldots, x_{j-1}, x_j^t, x_{j+1}, \ldots, x_n$ is a regular sequence on M, then $x_1, \ldots, x_n$ is: we may change the exponents to 1 one at a time. The issue may be considered mod $(x_1, \ldots, x_{j-1})M$. Therefore, it suffices to consider the case $j = 1$, and we need only show that if $x_1^t, x_2, \ldots, x_n$ is a regular sequence on M then so is $x_1, \ldots, x_n$. It is clear that if x_1^t is a nonzerodivisor then so is x_1.

By induction on n we may assume that $x_1, \ldots, x_{n-1}$ is a regular sequence on M. We need to show that if $x_nu \in (x_1, \ldots, x_{n-1})M$, then $u \in (x_1, x_2, \ldots, x_{n-1})M$. If we multiply by x_1^{t-1}, we find that

$$x_n(x_1^{t-1}u) \in (x_1^t, x_2, \ldots, x_{n-1})M,$$

and so

$$x_1^{t-1}u = x_1^tv_1 + x_2v_2 + \cdots + x_{n-1}v_{n-1},$$

i.e.,

$$x_1^{t-1}(u - x_1v_1) - x_2v_2 - \cdots - x_{n-1}v_{n-1} = 0.$$

By the induction hypothesis, $x_1, \ldots, x_{n-1}$ is a regular sequence on M, and by the first part, $x_1^{t-1}, x_2, ..., x_{n-1}$ is a regular sequence on M. By the Lemma on p. 1, we have that

$$u - x_1v_1 \in (x_1^{t-1}, x_2, \ldots, x_{n-1})M,$$

and so $u \in (x_1, \ldots, x_{n-1})M$, as required. $\square$

Theorem. *Let $x_1, \dots, x_n$ be a regular sequence on the R-module M, and let I denote the ideal $(x_1, \dots, x_n)R$. Let $a_1, \dots, a_n$ be nonnegative integers, and suppose that $u, u_1, \dots, u_n$ are elements of M such that*

$$(\#) \quad x_1^{a_1} \cdots x_n^{a_n} u = \sum_{j=1}^{n} x_j^{a_j+1} u_j.$$

Then $u \in IM$.

Proof. We use induction on the number of nonzero a_j: we are done if all are 0. If $a_i > 0$, let y be $\Pi_{j \neq i} x_j^{a_j}$. Rewrite (#) as $\sum_{j \neq i} x_j^{a_j+1} u_j - x_i^{a_j}(yu - x_i u_i) = 0$. Since powers of the x_j are again regular, the Lemma on p. 1 yields that $yu - x_i u_i \in x_i^{a_i} M + (x_j^{a_j+1} : j \neq i)M$ and so $yu \in x_i M + (x_j^{a_j+1} : j \neq i)M$. Now $a_i = 0$ in the monomial y, and there is one fewer nonzero a_j. The desired result now follows from the induction hypothesis. $\square$

If I is an ideal of a ring R, we can form the *associated graded ring*

$$\mathrm{gr}_I(R) = R/I \oplus I/I^2 \oplus \cdots \oplus I^k/I^{k+1} \oplus \cdots,$$

an $\mathbb{N}$-graded ring whose kth graded piece is I^k/I^{k+1}. If $f \in I^h$ represents an element $a \in I^h/I^{h+1} = [\mathrm{gr}_I R]_h$ and $g \in I^k$ represents an element $b \in I^k/I^{k+1} = [\mathrm{gr}_I(R)]_k$, then ab is the class of fg in I^{h+k}/I^{h+k+1}. Likewise, if M is an R-module, we can form

$$\mathrm{gr}_I M = M/IM \oplus IM/I^2M \oplus \cdots \oplus I^kM/I^{k+1}M \oplus \cdots.$$

This is an $\mathbb{N}$-graded module over $\mathrm{gr}_I(R)$ in an obvious way: with f and a as above, if $u \in I^kM$ represents an element $z \in I^kM/I^{k+1}M$, then the class of fu in $I^{h+k}M/I^{h+k+1}M$ represents az.

If $x_1, \dots, x_n \in R$ generate I, the classes $[x_i] \in I/I^2$ generate $\mathrm{gr}_I(R)$ as an (R/I)-algebra. Let $\theta : (R/I)[X_1, \dots, X_n] \twoheadrightarrow \mathrm{gr}_I(R)$ be the (R/I)-algebra map such that $X_i \mapsto [x_i]$. This is a surjection of graded (R/I)-algebras. By restriction of scalars, $\mathrm{gr}_I(M)$ is also a module over $(R/I)[X_1, \dots, X_n]$. The (R/I)-linear map $M/IM \hookrightarrow \mathrm{gr}_I M$ then gives a map

$$\theta_M : (R/I)[X_1, \dots, X_n] \otimes_{R/I} M/IM \to \mathrm{gr}_I(M).$$

Note that $\theta_R = \theta$. If $u \in M$ represents $[u]$ in M/IM and $t_1, \dots, t_n$ are nonnegative integers whose sum is k, then

$$X_1^{t_1} \cdots X_n^{t_n} \otimes [u] \mapsto [x_1^{t_1} \cdots x_n^{t_n} u],$$

where the right hand side is to be interpreted in $I^kM/I^{k+1}M$. Note that θ_M is surjective.

Theorem. *Let $x_1, \dots, x_n$ be a regular sequence on the R-module M, and suppose that $I = (x_1, \dots, x_n)R$. Let $X_1, \dots, X_n$ be indeterminates over the ring R/I. Then*

$$\mathrm{gr}_I(M) \cong (R/I)[X_1, \dots, X_n] \otimes_{R/I} (M/IM)$$

in such a way that the action of $[x_i] \in I/I^2 = [\mathrm{gr}_I(R)]_1$ on $\mathrm{gr}_I(M)$ is the same as multiplication by the variable X_i.

In particular, if $x_1, \dots, x_n$ is a regular sequence in R, then $\mathrm{gr}_I(R) \cong (R/I)[X_1, \dots, X_n]$ in such a way that $[x_i]$ corresponds to X_i.

In other words, if $x_1, \dots, x_n$ is a regular sequence on M (respectively, R), then the map θ_M (respectively, θ) discussed in the paragraph above is an isomorphism.

Proof. The issue is whether θ_M is injective. If not, there is a nontrivial relation on the monomials in the elements $[x_i]$ with coefficients in M/IM, and then there must be such a relation that is homogeneous of, say, degree k. Lifting to M, we see that this means that there is an $(M - IM)$-linear combination of mutually distinct monomials of degree k in $x_1, \dots, x_n$ which is in $I^{k+1}M$. Choose one monomial term in this relation: it will have the form $x_1^{a_1} \cdots x_n^{a_n} u$, where the sum of the a_j is k and $u \in M - IM$. The other monomials of degree k in the elements $x_1, \dots, x_n$ and the monomial generators of I^{k+1} all have as a factor at least one of the terms $x_1^{a_1+1}, \dots, x_n^{a_n+1}$. This yields that

$$(\#) \quad (\Pi_j x_j^{a_j})u = \sum_{j=1}^{n} x_j^{a_j+1} u_j.$$

By the preceding Theorem, $u \in IM$, contradictioning that $u \in M - IM$. $\square$

Cohen-Macaulay rings in the graded and local cases

We want to put special emphasis on the graded case for several reasons. One is its importance in projective geometry. Beyond that, there are many theorems about the graded case that make it easier both to understand and to do calculations. Moreover, many of the most important examples of Cohen-Macaulay rings are graded.

We first note:

Proposition. *Let M be an $\mathbb{N}$-graded or $\mathbb{Z}$-graded module over an $\mathbb{N}$-graded or $\mathbb{Z}$-graded Noetherian ring S. Then every associated prime of M is homogeneous. Hence, every minimal prime of the support of M is homogeneous and, in particular the associated (hence, the minimal) primes of S are homogeneous.*

Proof. Any associated prime P of M is the annihilator of some element u of M, and then every nonzero multiple of $u \neq 0$ can be thought of as a nonzero element of $S/P \cong Su \subseteq M$, and so has annihilator P as well. If u_i is a nonzero homogeneous component of u of degree i, its annihilator J_i is easily seen to be a homogeneous ideal of S. If $J_h \neq J_i$ we can choose a form F in one and not the other, and then Fu is nonzero with fewer homgeneous components then u. Thus, the homogeneous ideals J_i are all equal to, say, J, and clearly $J \subseteq P$. Suppose that $s \in P - J$ and subtract off all components of S that are in J, so that no nonzero component is in J. Let $s_a \notin J$ be the lowest degree component of s and u_b be the lowest degree component in u. Then $s_a\, u_b$ is the only term of degree $a + b$ occurring in $su = 0$, and so must be 0. But then $s_a \in \mathrm{Ann}_S\, u_b = J_b = J$, a contradiction. □

Corollary. *Let K be a field and let R be a finitely generated $\mathbb{N}$-graded K-algebra with $R_0 = K$. Let $\mathcal{M} = \bigoplus_{d=1}^{\infty} R_j$ be the homogeneous maximal ideal of R. Then $\dim(R) = \mathrm{height}\,(\mathcal{M}) = \dim(R_{\mathcal{M}})$.*

Proof. The dimension of R will be equal to the dimension of R/P for one of the minimal primes P of R. Since P is minimal, it is an associated prime and therefore is homogenous. Hence, $P \subseteq \mathcal{M}$. The domain R/P is finitely generated over K, and therefore its dimension is equal to the height of every maximal ideal including, in particular, $\mathcal{M}/P$. Thus,

$$\dim(R) = \dim(R/P) = \dim\big((R/P)_{\mathcal{M}}\big) \le \dim R_{\mathcal{M}} \le \dim(R),$$

and so equality holds throughout, as required. □

Proposition (homogeneous prime avoidance). *Let R be an $\mathbb{N}$-graded algebra, and let I be a homogeneous ideal of R whose homogeneous elements have positive degree. Let $P_1, \ldots, P_k$ be prime ideals of R. Suppose that every homogeneous element $f \in I$ is in $\bigcup_{i=1}^{k} P_i$. Then $I \subseteq P_j$ for some j, $1 \le j \le k$.*

Proof. We have that the set H of homogeneous elements of I is contained in $\bigcup_{i=1}^{k} P_k$. If $k = 1$ we can conclude that $I \subseteq P_1$. We use induction on k. Without loss of generality, we may assume that H is not contained in the union of any $k - 1$ if the P_j. Hence, for every i there is a homogeous element $g_i \in I$ that is not in any of the P_j for $j \neq i$, and so it must be in P_i. We shall show that if $k > 1$ we have a contradiction. By raising the g_i to suitable positive powers we may assume that they all have the same degree. Then $g_1^{k-1} + g_2 \cdots g_k \in I$ is a homogeneous element of I that is not in any of the P_j: g_1 is not in P_j for $j > 1$ but is in P_1, and $g_2 \cdots g_k$ is in each of $P_2, \ldots, P_k$ but is not in P_1. □

Now suppose that R is a finitely generated $\mathbb{N}$-graded algebra over $R_0 = K$, where K is a field. By a *homogenous system of parameters* for R we mean a sequence of homogeneous elements $F_1, \ldots, F_n$ of positive degree in R such that $n = \dim(R)$ and $R/F_1, \ldots, F_n)$ has Krull dimension 0. When R is a such a graded ring, a homogeneous system of parameters always exists. By homogeneous prime avoidance, there is a form F_1 that is not in the union of the minimal primes of R. Then $\dim(R/F_1) = \dim(R) - 1$. For the inductive step, choose forms of positive degree $F_2, \ldots, F_n$ whose images in R/F_1R are a homogeneous system of parameters for R/F_1R. Then $F_1, \ldots, F_n$ is a homogeneous system of parameters for R. □

Moreover, we have:

Theorem. *Let R be a finitely generated $\mathbb{N}$-graded K-algebra with $R_0 = K$ such that $\dim(R) = n$. A homogeneous system of parameters $F_1, \ldots, F_n$ for R always exists. Moreover, if $F_1, \ldots, F_n$ is a sequence of homogeneous elements of positive degree, then the following statements are equivalent.*

(1) *$F_1, \ldots, F_n$ is a homogeneous system of parameters.*

(2) *m is nilpotent modulo $(F_1, \ldots, F_n)R$.*

(3) *$R/(F_1, \ldots, F_n)R$ is finite-dimensional as a K-vector space.*

(4) *R is module-finite over the subring $K[F_1, \ldots, F_n]$.*

Moreover, when these conditions hold, $F_1, \ldots, F_n$ are algebraically independent over K, so that $K[F_1, \ldots, F_n]$ is a polynomial ring.

Proof. We have already shown existence.

(1) $\Rightarrow$ (2). If $F_1, \ldots, F_n$ is a homogeneous system of parameters, we have that

$$\dim\bigl(R/F_1, \ldots, F_n)\bigr) = 0.$$

We then know that all prime ideals are maximal. But we know as well that the maximal ideals are also minimal primes, and so must be homogeneous. Since there is only one homogenous maximal ideal, it must be $m/(F_1, \ldots, F_n)R$, and it follows that m is nilpotent on $(F_1, \ldots, F_n)R$.

(2) $\Rightarrow$ (3). If m is nilpotent modulo $(F_1, \ldots, F_n)R$, then the homogeneous maximal ideal of $\overline{R} = R/(F_1, \ldots, F_n)R$ is nilpotent, and it follows that $[\overline{R}]_d = 0$ for all $d \gg 0$. Since each $\overline{R}_d$ is a finite dimensional vector space over K, it follows that $\overline{R}$ itself is finite-dimensional as a K-vector space.

(3) $\Rightarrow$ (4). This is immediate from the homogeneous form of Nakayama's Lemma: a finite set of homogeneous elements of R whose images in $\overline{R}$ are a K-vector space basis will span R over $K[F_1, \ldots, F_n]$, since the homogenous maximal ideal of $K[F_1, \ldots, F_n]$ is generated by $F_1, \ldots, F_n$.

(4) $\Rightarrow$ (1). If R is module-finite over $K[F_1, \ldots, F_n]$, this is preserved mod $(F_1, \ldots, F_n)$, so that $R/(F_1, \ldots, F_n)$ is module-finite over K, and therefore zero-dimensional as a ring.

Finally, when R is a module-finite extension of $K[F_1, \ldots, F_n]$, the two rings have the same dimension. Since $K[F_1, \ldots, F_n]$ has dimension n, the elements $F_1, \ldots, F_n$ must be algebraically independent. $\square$

The technique described in the discussion that follows is very useful both in the local and graded cases.

Discussion: making a transition from one system of parameters to another. Let R be a Noetherian ring of Krull dimension n, and assume that one of the two situations described below holds.

(1) (R, m, K) is local and $f_1, \ldots, f_n$ and $g_1, \ldots, g_n$ are two systems of parameters.

(2) R is finitely generated $\mathbb{N}$-graded over $R_0 = K$, a field, m is the homogeneous maximal ideal, and $f_1, \ldots, f_n$ and $g_1, \ldots, g_n$ are two homogeneous systems of parameters for R.

We want to observe that in this situation there is a finite sequence of systems of parameters (respectively, homogeneous systems of parameters in case (2)) starting with $f_1, \ldots, f_n$ and ending with $g_1, \ldots, g_n$ such that any two consecutive elements of the sequence agree in all but one element (i.e., after reordering, only the i th terms are possibly different for a single value of i, $1 \leq i \leq n$). We can see this by induction on n. If $n = 1$ there is nothing to prove. If $n > 1$, first note that we can choose h (homogeneous of positive degree in the graded case) so as to avoid all minimal primes of $(f_2, \ldots, f_n)R$ and all minimal primes of $(g_2, \ldots, g_n)R$. Then it suffices to get a sequence from $h, f_2, \ldots, f_n$ to $h, g_2, \ldots, g_n$, since the former differs from $f_1, \ldots, f_n$ in only one term and the latter differs from $g_1, \ldots, g_n$ in only one term. But this problem can be solved by working in R/hR and getting a sequence from the images of $f_2, \ldots, f_n$ to the images of $g_2, \ldots, g_n$, which we can do by the induction hypothesis. We lift all of the systems of parameters back to R by taking, for each one, h and inverse images of the elements in the sequence in R (taking a homogeneous inverse image in the graded case), and always taking the same inverse image for each element of R/hR that occurs. $\square$

The following result now justifies several assertions about Cohen-Macaulay rings made without proof earlier.

Note that a regular sequence in the maximal ideal of a local ring (R, m, K) is always part of a system of parameters: each element is not in any associated prime of the ideal generated by its predecessors, and so cannot be any minimal primes of that ideal. It follows that as we kill successive elements of the sequence, the dimension of the quotient drops by one at every step.

Corollary. *Let (R, m, K) be a local ring. There exists a system of parameters that is a regular sequence if and only if every system of parameters is a regular sequence. In this case, for every prime ideal I of R of height k, there is a regular sequence of length k in I.*

Moreover, for every prime ideal P of R, R_P also has the property that every system of parameters is a regular sequence.

Proof. For the first statement, we can choose a chain as in the comparison statement just above. Thus, we can reduce to the case where the two systems of parameters differ in only one element. Because systems of parameters are permutable and regular sequences are permutable in the local case, we may assume that the two systems agree except possibly for the last element. We may therefore kill the first $\dim(R) - 1$ elements, and so reduce to the case where x and y are one element systems of parameters in a local ring R of dimension 1. Then x has a power that is a multiple of y, say $x^h = uy$, and y has a power that is a multiple of x. If x is not a zerodivisor, neither is x^h, and it follows that y is not a zerodivisor. The converse is exactly similar.

Now suppose that I is any ideal of height h. Choose a maximal sequence of elements (it might be empty) of I that is part of a system of parameters, say $x_1, \ldots, x_k$. If $k < h$, then I cannot be contained in the union of the minimal primes of $(x_1, \ldots, x_k)$: otherwise, it will be contained in one of them, say Q, and the height of Q is bounded by k. Chose $x_{k+1} \in I$ not in any minimal prime of $(x_1, \ldots, x_k)R$. Then $x_1, \ldots, x_{k+1}$ is part of a system of parameters for R, contradicting the maximality of the sequence $x_1, \ldots, x_k$.

Finally, consider the case where $I = P$ is prime. Then P contains a regular sequence $x_1, \ldots, x_k$, which must also be regular in R_P, and, hence, part of a system of parameters. Since $\dim(R_P) = k$, it must be a system of parameters. $\square$

Lemma. *Let K be a field and assume either that*

(1) *R is a regular local ring of dimension n and $x_1, \ldots, x_n$ is a system of parameters*

or

(2) *$R = K[x_1, \ldots, x_n]$ is a graded polynomial ring over K in which each of the x_i is a form of positive degree.*

Let M be a nonzero finitely generated R-module which is $\mathbb{Z}$-graded in case (2). Then M is free if and only if $x_1, \ldots, x_n$ is a regular sequence on M.

Proof. The "only if" part is clear, since $x_1, \ldots, x_n$ is a regular sequence on R and M is a direct sum of copies of R. Let $m = (x_1, \ldots, x_n)R$. Then $V = M/mM$ is a finite-dimensional K-vector space that is graded in case (2). Choose a K-vector space basis for V consisting of homogeneous elements in case (2), and let $u_1, \ldots, u_h \in M$ be elements of M that lift these basis elements and are homogeneous in case (2). Then the u_j span M by the relevant form of Nakayama's Lemma, and it suffices to prove that they have no nonzero relations over R. We use induction on n. The result is clear if $n = 0$.

Assume $n > 0$ and let $N = \{(r_1, \ldots, r_h) \in R^h : r_1u_1 + \cdots + r_hu_h = 0\}$. By the induction hypothesis, the images of the u_j in M/x_1M are a free basis for M/x_1M. It follow that if $\rho = (r_1, \ldots, r_h) \in N$, then every r_j is 0 in R/x_1R, i.e., that we can write $r_j = x_1s_j$ for all j. Then $x_1(s_1u_1 + \cdots + s_hu_h) = 0$, and since x_1 is not a zerodivisor on M, we have that $s_1u_1 + \cdots + s_hu_h = 0$, i.e., that $\sigma = (s_1, \ldots, s_h) \in N$. Then $\rho = x_1\sigma \in x_1N$, which shows that $N = x_1N$. Thus, $N = 0$ by the appropriate form of Nakayama's Lemma. $\square$

We next observe:

Theorem. *Let R be a finitely generated graded algebra of dimension n over $R_0 = K$, a field. Let m denote the homogeneous maximal ideal of R. The following conditions are equivalent.*

(1) *Some homogeneous system of parameters is a regular sequence.*

(2) *Every homogeneous system of parameters is a regular sequence.*

(3) *For some homogeneous system of parameters* $F_1, \dots, F_n$*, R is a free-module over* $K[F_1, \dots, F_n]$.

(4) *For every homogeneous system of parameters* $F_1, \dots, F_n$*, R is a free-module over* $K[F_1, \dots, F_n]$.

(5) R_m *is Cohen-Macaulay.*

(6) R *is Cohen-Macaulay.*

Proof. The proof of the equivalence of (1) and (2) is the same as for the local case, already given above.

The preceding Lemma yields the equivalence of (1) and (3), as well as the equivalence of (2) and (4). Thus, (1) through (4) are equivalent.

It is clear that (6) $\Rightarrow$ (5). To see that (5) $\Rightarrow$ (2) consider a homogeneous system of parameters in R. It generates an ideal whose radical is m, and so it is also a system of parameters for R_m. Thus, the sequence is a regular sequence in R_m. We claim that it is also a regular sequence in R. If not, x_{k+1} is contained in an associated prime of $(x_1, \dots, x_k)$ for some k, $0 \le k \le n-1$. Since the associated primes of a homogeneous ideal are homogeneous, this situation is preserved when we localize at m, which gives a contradiction.

To complete the proof, it will suffice to show that (1) $\Rightarrow$ (6). Let $F_1, \dots, F_n$ be a homogeneous system of parameters for R. Then R is a free module over $A = K[F_1, \dots, F_n]$, a polynomial ring. Let Q be any maximal ideal of R and let P denote its contraction to A, which will be maximal. These both have height n. Then $A_P \to R_Q$ is faithfully flat. Since A is regular, A_P is Cohen-Macaulay. Choose a system of parameters for A_P. These form a regular sequence in A_P, and, hence, in the faithfully flat extension R_Q. It follows that R_Q is Cohen-Macaulay. $\square$

From part (2) of the Lemma on p. 8 we also have:

Theorem. *Let R be a module-finite local extension of a regular local ring A. Then R is Cohen-Macaulay if and only if R is A-free.*

It it is not always the case that a local ring (R, m, K) is module-finite over a regular local ring in this way. But it does happen frequently in the complete case. Notice that the property of being a regular sequence is preserved by completion, since the completion $\widehat{R}$ of a local ring is faithfully flat over R, and so is the property of being a system of parameters. Hence, R is Cohen-Macaulay if and only if $\widehat{R}$ is Cohen-Macaulay.

If R is complete and contains a field, then there is a coefficient field for R, i.e., a field $K \subseteq R$ that maps isomorphically onto the residue class field K of R. Then, if $x_1, \dots, x_n$ is a system of parameters, R turns out to be module-finite over the formal power series ring $K[[x_1, \dots, x_n]]$ in a natural way. Thus, in the complete equicharacteristic local case, we can always find a regular ring $A \subseteq R$ such that R is module-finite over A, and think of the Cohen-Macaulay property as in the Theorem above.

The structure theory of complete local rings is discussed in detail in the Lecture Notes from Math 615, Winter 2007: see the Lectures of March 21, 23, 26, 28, and 30 as well as the Lectures of April 2 and April 4.

Cohen-Macaulay modules

All of what we have said about Cohen-Macaulay rings generalizes to a theory of Cohen-Macaulay modules. We give a few of the basic definitions and results here: the proofs are very similar to the ring case, and are left to the reader.

If M is a module over a ring R, the *Krull dimension* of M is the Krull dimension of $R/\mathrm{Ann}_R(I)$. If (R, m, K) is local and $M \neq 0$ is finitely generated of Krull dimension d, a *system of parameters* for M is a sequence of elements $x_1, \dots, x_d \in m$ such that, equivalently:

(1) $\dim\bigl(M/(x_1, \dots, x_d)M\bigr) = 0$.

(2) The images of $x_1, \dots, x_d$ form a system of parameters in $R/\mathrm{Ann}_R M$.

In this local situation, M is *Cohen-Macaulay* if one (equivalently, every) system of parameters for M is a regular sequence on M. If J is an ideal of $R/\mathrm{Ann}_R M$ of height h, then it contains part of a system of parameters for $R/\mathrm{Ann}_R M$ of height h, and this will be a regular sequence on M. It follows that the Cohen-Macaulay property for M passes to M_P for every prime P in the support of M. The arguments are all essentially the same as in the ring case.

If R is any Noetherian ring $M \neq 0$ is any finitely generated R-module, M is called *Cohen-Macaulay* if all of its localizations at maximal (equivalently, at prime) ideals in its support are Cohen-Macaulay.

The Cohen-Macaulay condition is increasingly restrictive as the Krull dimension increases. In dimension 0, every local ring is Cohen-Macaulay. In dimension one, it is sufficient, but not necessary, that the ring be reduced: the precise characterization in dimension one is that the maximal ideal not be an embbeded prime ideal of (0). Note that $K[[x, y]]/(x^2)$ *is* Cohen-Macaulay, while $K[[x, y]]/(x^2, xy)$ is not. Also observe that all one-dimensional domains are Cohen-Macaulay.

In dimension 2, it suffices, but is not necessary, that the ring R be normal, i.e., integrally closed in its ring of fractions. Note that a normal Noetherian ring is a finite product of normal domains. If (R, m, K) is local and normal, then it is a doman. The associated primes of a principal ideal are minimal if R is normal. Hence, if x, y is a system of parameters, y is not in any associated prime of xR, i.e., it is not in any associated prime of the module R/xR, and so y is not a zerodivisor modulo xR.

The two dimensional domains $K[[x^2, x^2, y, xy]]$ and $K[x^4, x^3y, xy^3, y^4]]$ (one may also use single brackets) are not Cohen-Macaulay: as an exercise, the reader may try to see that y is a zerodivisor mod x^2 in the first, and that y^4 is a zerodivisor mod x^4 in the second. On the other hand, while $K[[x^2, x^3, y^2, y^3]]$ is not normal, it is Cohen-Macaulay.

Segre products

Let R and S be finitely generated $\mathbb{N}$-graded K-algebras with $R_0 = S_0 = K$. We define the *Segre product* $R \circledS_K S$ of R and S over K to be the ring

$$\bigoplus_{n=1}^{\infty} R_n \otimes_K S_n,$$

which is a subring of $R \otimes_K S$. In fact, $R \otimes_K S$ has a grading by $\mathbb{N} \times \mathbb{N}$ whose (m, n) component is $R_m \otimes_K S_n$. (There is no completely standard notation for Segre products: the one used here is only one possbility.) The vector space

$$\bigoplus_{m \neq n} R_m \otimes_K S_n \subseteq R \otimes_K S$$

is an $R \circledS_K S$-submodule of $R \otimes_K S$ that is an $R \circledS_K S$-module complement for $R \circledS_K S$. That is, $R \circledS_K S$ is a direct summand of $R \otimes_K S$ when the latter is regarded as an $R \circledS_K S$-module. It follows that $R \circledS_K S$ is Noetherian and, hence, finitely generated over K. Moreover, if $R \otimes_K S$ is normal then so is $R \circledS_K S$. In particular, if R is normal and S is a polynomial ring over K then $R \circledS_K S$ is normal.

Let $S = K[X, Y, Z]/(X^3 + Y^3 + Z^3) = K[x, y, z]$, where K is a field of characteristic different from 3: this is a homogeneous coordinate ring of an elliptic curve C, and is often referred to as a *cubical cone*. Let $T = K[s, t]$, a polynomial ring, which is a homogeneous coordinate ring for the projective line $\mathbb{P}^1 = \mathbb{P}^1_K$. The Segre product of these two rings is $R = K[xs, ys, zs, xt, yt, zt] \subseteq S[s, t]$, which is a homogeneous coordinate ring for the smooth projective variety $C \times \mathbb{P}^1$. This ring is a normal domain with an isolated singularity at the origin: that is, its localization at any prime ideal except the homogeneous maximal ideal m is regular. R and R_m are normal but not Cohen-Macaulay.

We give a proof that R is not Cohen-Macaulay. The equations

$$(zs)^3 + \big((xs)^3 + (ys)^3\big) = 0 \quad \text{and} \quad (zt)^3 + \big((xt)^3 + (yt)^3\big) = 0$$

show that zs and zt are both integral over $D = K[xs, ys, xt, zt] \subseteq R$. The elements x, y, s, and t are algebraically independent, and the fraction field of D is $K(xs, ys, t/s)$, so that $\dim(D) = 3$, and

$$D \cong K[X_{11}, X_{12}, X_{21}, X_{22}]/(X_{11}X_{22} - X_{12}X_{21})$$

with X_{11}, X_{12}, X_{21}, X_{22} mapping to xs, ys, xt, yt respectively.

It is then easy to see that ys, xt, $xs - yt$ is a homogeneous system of parameters for D, and, consequently, for R as well. The relation

$$(zs)(zt)(xs - yt) = (zs)^2(xt) - (zt)^2(ys)$$

now shows that R is *not* Cohen-Macaulay, for $(zs)(zt) \notin (xt, ys)R$. To see this, suppose otherwise. The map

$$K[x, y, z, s, t] \to K[x, y, z]$$

that fixes $K[x, y, z]$ while sending $s \mapsto 1$ and $t \mapsto 1$ restricts to give a K-algebra map

$$K[xs, ys, zs, xt, yt, zt] \to K[x, y, z].$$

If $(zs)(zt) \in (xt, ys)R$, applying this map gives $z^2 \in (x, y)K[x, y, z]$, which is false — in fact, $K[x, y, z]/(x, y) \cong K[z]/(z^3)$. $\square$

Fibers

Let $f : R \to S$ be a ring homomorphism and let P be a prime ideal of R. We write κ_P for the canonically isomorphic R-algrebras

$$\mathrm{frac}\,(R/P) \cong R_P/PR_P.$$

By the *fiber* of f over P we mean the κ_P-algebra

$$\kappa_P \otimes_R S \cong (R - P)^{-1}S/PS$$

which is also an R-algebra (since we have $R \to \kappa_P$) and an S-algebra. One of the key points about this terminology is that the map

$$\mathrm{Spec}\,(\kappa_P \otimes_R S) \to \mathrm{Spec}\,(S)$$

gives a bijection between the prime ideals of $\kappa_P \otimes_R S$ and the prime ideals of S that lie over $P \subseteq R$. In fact, it is straightforward to check that $\mathrm{Spec}\,(\kappa_P \otimes_R S)$ is homeomorphic with its image in $\mathrm{Spec}\,(S)$.

It is also said that $\mathrm{Spec}\,(\kappa_P \otimes_R S)$ is the *scheme-theoretic* fiber of the map

$$\mathrm{Spec}\,(S) \to \mathrm{Spec}\,(R).$$

This is entirely consistent with thinking of the fiber of a map of sets $g : Y \to X$ over a point $P \in X$ as

$$g^{-1}(P) = \{Q \in Y : g(Q) = P\}.$$

In our case, we may take $g = \mathrm{Spec}\,(f)$, $Y = \mathrm{Spec}\,(S)$, and $X = \mathrm{Spec}\,(R)$, and then $\mathrm{Spec}\,(\kappa_P \otimes_R S)$ may be naturally identified with the set-theoretic fiber of

$$\mathrm{Spec}\,(S) \to \mathrm{Spec}\,(R).$$

If R is a domain, the fiber over the prime ideal (0) of R, namely $\mathrm{frac}\,(R) \otimes_R S$, is called the *generic fiber* of $R \to S$.

If (R, m, K) is quasilocal, the fiber $K \otimes_R S = S/mS$ over the unique closed point m of $\mathrm{Spec}\,(R)$ is called the *closed fiber* of $R \to S$.

Proposition. *Let* $(R,\, m,\, K) \to (S,\, Q,\, L)$ *be a flat local homomorphism of local rings. Then*

(a) $\dim(S) = \dim(R) + \dim(S/mS)$, *the sum of the dimensions of the base and of the closed fiber.*

(b) *If* R *is regular and* S/mS *is regular, then* S *is regular.*

Proof. (a) We use induction on $\dim(R)$. If $\dim(R) = 0$, m and mS are nilpotent. Then $\dim(S) = \dim(S/mS) = \dim(R) + \dim(S/mS)$, as required. If $\dim(R) > 0$, let J be the ideal of nilpotent elements in R. Then $\dim(R/J) = \dim(R)$, $\dim(S/JS) = \dim(S)$, and the closed fiber of $R/J \to S/JS$, which is still a flat and local homomorphism, is S/mS. Therefore, we may consider the map $R/J \to S/JS$ instead, and so we may assume that R is reduced. Since $\dim(R) > 0$, there is an element $f \in m$ not in any minimal prime of R, and, since R is reduced, f is not in any associated prime of R, i.e., f is a nonzerodivisor in R. Then the fact that S is flat over R implies that f is not a zerodivisor in S. We may apply the induction hypothesis to $R/fR \to S/fS$, and so

$$\dim(S) - 1 = \dim(S/fS) = \dim(R/f) + \dim(S/mS) = \dim(R) - 1 + \dim(S/mS),$$

and the result follows.

(b) The least number of generators of Q is at most the sum of the number of generators of m and the number of generators of Q/mS, i.e., it is bounded by $\dim(R)+\dim(S/mS) = \dim(S)$ by part (a). The other inequality always holds, and so S is regular. $\square$

Corollary. *Let* $R \to S$ *be a flat homomorphism of Noetherian rings. If* R *is regular and the fibers of* $R \to S$ *are regular, then* S *is regular.*

Proof. If Q is any prime of S we may apply part (b) of the preceding Theorem, since S_Q/PS_Q is a localization of the fiber $\kappa_P \otimes_R S$, and therefore regular. $\square$

Catenary and universally catenary rings

A Noetherian ring is called *catenary* if for any two prime ideals $P \subseteq Q$, any two saturated chains of primes joining P to Q have the same length. In this case, the common length will be the same as the dimension of the local domain R_Q/PR_Q.

Nagata was the first to give examples of Notherian rings that are not catenary. E.g., in [M. Nagata, *Local Rings*, Interscience, New York, 1962] Appendix, pp. 204–5, Nagata gives an example of a local domain $(D,\, m)$ of dimension 3 containing a height one prime P such that $\dim(D/P) = 1$, so that $(0) \subset Q \subset m$ is a saturated chain, while the longest saturated chains joining (0) to m have the form $(0) \subset P_1 \subset P_2 \subset m$. One has to work hard to construct Noetherian rings that are not catenary. Nagata also gives an example of a ring R that is catenary, but such that $R[x]$ is not catenary.

Notice that a localization or homomorphic image of a catenary ring is automatically catenary.

R is called *universally catenary* if every polynomial ring over R is catenary. This implies that every ring essentially of finite type over R is catenary.

A very important fact about Cohen-Macaulay rings is that they are catenary. Moreover, a polynomial ring over a Cohen-Macaulay ring is again a Cohen-Macaulay ring, which then implies that every Cohen-Macaulay ring is universally catenary. In particular, regular rings are universally catenary. Cohen-Macaulay local rings have a stronger property: they are equidimensional, and all saturated chains from a minimal prime to the maximal ideal have length equal to the dimension of the local ring.

We shall prove the statements in the paragraph above. We first note:

Theorem. *If R is Cohen-Macaulay, so is the polynomial ring in n variables over R.*

Proof. By induction, we may assume that $n = 1$. Let $\mathcal{M}$ be a maximal ideal of $R[X]$ lying over m in R. We may replace R by R_m and so we may assume that (R, m, K) is local. Then $\mathcal{M}$, which is a maximal ideal of $R[x]$ lying over m, corresponds to a maximal ideal of $K[x]$: each of these is generated by a monic irreducible polynomial f, which lifts to a monic polynomial F in $R[x]$. Thus, we may assume that $\mathcal{M} = mR[x] + FR[X]$. Let $x_1, \ldots, x_d$ be a system of parameters in R, which is also a regular sequence. We may kill the ideal generated by these elements, which also form a regular sequence in $R[X]_{\mathcal{M}}$. We are now in the case where R is an Artin local ring. It is clear that the height of $\mathcal{M}$ is one. Because F is monic, it is not a zerodivisor: a monic polynomial over any ring is not a zerodivisor. This shows that the depth of $\mathcal{M}$ is one, as needed. $\square$

Theorem. *Let (R, m, K) be a local ring and $M \neq 0$ a finitely generated Cohen-Macaulay R-module of Krull dimension d. Then every nonzero submodule N of M has Krull dimension d.*

Proof. We replace R by $R/\mathrm{Ann}_R M$. Then every system of parameters for R is a regular sequence on M. We use induction on d. If $d = 0$ there is nothing to prove. Assume $d > 0$ and that the result holds for smaller d. If M has a submodule $N \neq 0$ of dimension $\leq d - 1$, we may choose N maximal with respect to this property. If N' is any nonzero submodule of M of dimension $< d$, then $N' \subseteq N$. To see this, note that $N \oplus N'$ has dimension $< d$, and maps onto $N + N' \subseteq M$, which therefore also has dimension $< d$. By the maximality of N, we must have $N + N' = N$. Since M is Cohen-Macaulay and $d \geq 1$, we can choose $x \in m$ not a zerodivisor on M, and, hence, also not a zerodivisor on N. We claim that x is not a zerodivisor on $\overline{M} = M/N$, for if $u \in M - N$ and $xu \in N$, then $Rxu \subseteq N$ has dimension $< d$. But this module is isomorphic with $Ru \subseteq M$, since x is not a zerodivisor, and so $\dim(Ru) < d$. But then $Ru \subseteq N$. Consequently, multiplication by x induces an isomorphism of the exact sequence $0 \to N \to M \to \overline{M} \to 0$ with the sequence $0 \to xN \to xM \to x\overline{M} \to 0$, and so this sequence is also exact. But we have a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N & \longrightarrow & M & \longrightarrow & \overline{M} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & xN & \longrightarrow & xM & \longrightarrow & x\overline{M} & \longrightarrow & 0
\end{array}$$

where the vertical arrows are inclusions. By the nine lemma, or by an elementary diagram chase, the sequence of cokernels $0 \to N/xN \to M/xM \to \overline{M}/x\overline{M} \to 0$ is exact. Because x is not a zerodivisor on M, it is part of a system of parameters for R, and can be extended to a system of parameters of length d, which is a regular sequence on M. Since x is a nonzerodivisor on N and M, $\dim(N/xN) = \dim(N) - 1 < d - 1$, while M/xM is Cohen-Macaulay of dimension $d - 1$. This contradicts the induction hypothesis. $\square$

Corollary. *If (R, m, K) is Cohen-Macaulay, R is equidimensional: every minimal prime $\mathfrak{p}$ is such that* $\dim(R/\mathfrak{p}) = \dim(R)$.

Proof. If $\mathfrak{p}$ is minimal, it is an associated prime of R, and we have $R/\mathfrak{p} \hookrightarrow R$. Since all nonzero submodules of R have dimension $\dim(R)$, the result follows. $\square$

Thus, a Cohen-Macaulay local ring cannot exhibit the kind of behavior one observes in $R = K[[x, y, z]]/\big((x, y) \cap (z)\big)$: this ring has two minimal primes. One of them, $\mathfrak{p}_1$, generated by the images of x and y, is such that $R/\mathfrak{p}_1$ has dimension 1. The other, $\mathfrak{p}_2$, generated by the image of z, is such that $R/\mathfrak{p}_2$ has dimension 2. Note that while R is not equidimensional, it is still catenary.

We next observe:

Theorem. *In a Cohen-Macaulay ring R, if $P \subseteq Q$ are prime ideals of R then every saturated chain of prime ideals from P to Q has length* $\mathrm{height}(Q) - \mathrm{height}(P)$. *Thus, R is catenary.*

It follows that every ring essentially of finite type over a Cohen-Macaulay ring is universally catenary.

Proof. The issues are unaffected by localizing at Q. Thus, we may assume that R is local and that Q is the maximal ideal. There is part of a system of parameters of length $h = \mathrm{height}(P)$ contained in P, call it $x_1, \ldots, x_h$, by the Corollary near the bottom of p. 7 of the Lecture Notes of Septermber 5. This sequence is a regular sequence on R and so on R_P, which implies that its image in R_P is system of parameters. We now replace R by $R/(x_1, \ldots, x_h)$: when we kill part of a system of parameters in a Cohen-Macaulay ring, the image of the rest of that system of parameters is both a system of parameters and a regular sequence in the quotient. Thus, R remains Cohen-Macaulay. Q and P are replaced by their images, which have heights $\dim(R) - h$ and 0, and $\dim(R) - h = \dim\big(R/(x_1, \ldots, x_h)\big)$. We have therefore reduced to the case where (R, Q) is local and P is a minimal prime.

We know that $\dim(R) = \dim(R/P)$, and so at least one saturated chain from P to Q has length $\mathrm{height}\,(Q) - \mathrm{height}\,(P) = \mathrm{height}\,(Q) - 0 = \dim(R)$. To complete the proof, it will suffice to show that all saturated chains from P to Q have the same length, and we may use induction on $\dim(R)$. Consider two such chains, and let their smallest elements other than P be P_1 and P_1'. We claim that both of these are height one primes: if, say, P_1 is not height one we can localize at it and obtain a Cohen-Macaulay local ring (S, m) of dimension at least two and a saturated chain $\mathfrak{p} \subseteq m$ with $\mathfrak{p} = PS$ minimal in S. Choose an element $y \in m$ that is not in any minimal primes of S: its image will be a system of parameters for $S/\mathfrak{p}$, so that $Ry + \mathfrak{p}$ is m-primary. Extend y to a regular sequence of length two in S: the second element has a power of the form $ry + u$, so that $y,\, ry + u$ is a regular sequence, and, hence, so is $y,\, u$. But then $u,\, y$ is a regular sequence, a contradiction, since $u \in \mathfrak{p}$. Thus, P_1 (and, similarly, P_1'), have height one.

Choose an element f in P_1 not in any minimal prime of R, and an element g of P_1' not in any minimal prime of R. Then fg is a nonzerodivisor in R, and P_1, P_1' are both minimal primes of xy. The ring $R/(xy)$ is Cohen-Macaulay of dimension $\dim(R) - 1$. The result now follows from the induction hypothesis applied to $R/(xy)$: the images of the two saturated chains (omitting P from each) give saturated chains joining $P_1/(xy)$ (respectively, $P_1'/(xy)$) to $Q/(xy)$ in $R/(xy)$. These have the same length, and, hence, so did the original two chains.

The final statement now follows because a polynomial ring over a Cohen-Macaulay ring is again Cohen-Macaulay. $\square$

Note that one does not expect the completion of a local doman to be a domain, even when it is a localization of a ring finitely generated over the complex numbers. For example, consider the one-dimensional domain S $= \mathbb{C}[x,\, y]/(y^2 - x^2 - x^3)$. This is a domain because $x^2 + x^3$ is not a perfect square in $\mathbb{C}[x,\, y]$ (and, hence, not in its fraction field either, since $\mathbb{C}[x,\, y]$ is normal). If $m = (x,\, y)S$, then S_m is a local domain of dimension one. The completion of this ring is $\cong \mathbb{C}[[x,\, y]]/(y^2 - x^2 - x^3)$. This ring is not a domain: the point is that $x^2 + x^3 = x^2(1 + x)$ is a perfect square in the formal power series ring. Its square root may be written down explicitly using Newton's binomial theorem.

Flat base change and Hom

We want to discuss in some detail when a short exact sequence splits. The following result is very useful.

Theorem (Hom commutes with flat base change). *If S is a flat R-algebra and M, N are R-modules such that M is finitely presented over R, then the canonical homomorphism*

$$\theta_M \colon S \otimes_R \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_S(S \otimes_R M, S \otimes_R N)$$

sending $s \otimes f$ to $s(\mathbf{1}_S \otimes f)$ is an isomorphism.

Proof. It is easy to see that θ_R is an isomorphism and that $\theta_{M_1 \oplus M_2}$ may be identified with $\theta_{M_1} \oplus \theta_{M_2}$, so that θ_G is an isomorphism whenever G is a finitely generated free R-module.

Since M is finitely presented, we have an exact sequence $H \to G \twoheadrightarrow M \to 0$ where G, H are finitely generated free R-modules. In the diagram below the right column is obtained by first applying $S \otimes_R \_$ (exactness is preserved since $\otimes$ is right exact), and then applying $\mathrm{Hom}_S(\_, S \otimes_R N)$, so that the right column is exact. The left column is obtained by first applying $\mathrm{Hom}_R(\_, N)$, and then $S \otimes_R \_$ (exactness is preserved because of the hypothesis that S is R-flat). The squares are easily seen to commute.

$$\begin{array}{ccc}
S \otimes_R \mathrm{Hom}_R(H,N) & \xrightarrow{\theta_H} & \mathrm{Hom}_S(S \otimes_R H, S \otimes_R N) \\
\uparrow & & \uparrow \\
S \otimes_R \mathrm{Hom}_R(G,N) & \xrightarrow{\theta_G} & \mathrm{Hom}_S(S \otimes_R G, S \otimes_R N) \\
\uparrow & & \uparrow \\
S \otimes_R \mathrm{Hom}_R(M,N) & \xrightarrow{\theta_M} & \mathrm{Hom}_S(S \otimes_R M, S \otimes_R N) \\
\uparrow & & \uparrow \\
0 & \longrightarrow & 0
\end{array}$$

From the fact, established in the first paragraph, that θ_G and θ_H are isomorphisms and the exactness of the two columns, it follows that θ_M is an isomorphism as well (kernels of isomorphic maps are isomorphic). $\square$

Corollary. *If W is a multiplicative system in R and M is finitely presented, we have that* $W^{-1}\mathrm{Hom}_R(M,N) \cong \mathrm{Hom}_{W^{-1}R}(W^{-1}M, W^{-1}N)$.

Moreover, if (R, m) is a local ring and both M, N are finitely generated, we may identify $\mathrm{Hom}_{\widehat{R}}(\widehat{M}, \widehat{N})$ *with the m-adic completion of* $\mathrm{Hom}_R(M,N)$ *(since m-adic completion is the same as tensoring over R with $\widehat{R}$ (as covariant functors) on finitely generated R-modules).* $\square$

When does a short exact sequence split?

Throughout this section, $0 \to N \xrightarrow{\alpha} M \xrightarrow{\beta} Q \to 0$ is a short exact sequence of modules over a ring R. There is no restriction on the characteristic of R. We want to discuss the problem of when this sequence splits. One condition is that there exist a map $\eta : M \to N$ such that $\eta\alpha = \mathbf{1}_N$. Let $Q' = \mathrm{Ker}\,(\eta)$. Then Q' is disjoint from the image $\alpha(N) = N'$ of N in M, and $N' + Q' = M$. It follows that M is the internal direct sum of N' and Q' and that β maps Q' isomorphically onto Q.

Similarly, the sequence splits if there is a map $\theta : Q \to M$ such that $\beta\theta = \mathbf{1}_Q$. In this case let $N' = \alpha(N)$ and $Q' = \theta(Q)$. Again, N' and Q' are disjoint, and $N' + Q' = M$, so that M is again the internal direct sum of N' and Q'.

Proposition. *Let R be an arbitrary ring and let*

$$(\#) \quad 0 \to N \xrightarrow{\alpha} M \xrightarrow{\beta} Q \to 0$$

be a short exact sequence of R-modules. Consider the sequence

$$(*) \quad 0 \to \mathrm{Hom}_R(Q,\, N) \xrightarrow{\alpha_*} \mathrm{Hom}_R(Q,\, M) \xrightarrow{\beta_*} \mathrm{Hom}_R(Q,\, Q) \to 0$$

which is exact except possibly at $\mathrm{Hom}_R(Q,\, Q)$, and let $C = \mathrm{Coker}\,(\beta_*)$. The following conditions are equivalent:

(1) *The sequence $(\#)$ is split.*

(2) *The sequence $(*)$ is exact.*

(3) *The map β_* is surjective.*

(4) $C = 0$.

(5) *The element $\mathbf{1}_Q$ is in the image of β_*.*

Proof. Because Hom commutes with finite direct sum, we have that (1) $\Rightarrow$ (2), while (2) $\Rightarrow$ (3) $\Leftrightarrow$ (4) $\Rightarrow$ (5) is clear. It remains to show that (5) $\Rightarrow$ (1). Suppose $\theta : Q \to M$ is such that $\beta_*(\theta) = \mathbf{1}_Q$. Since β_* is induced by composition with β, we have that $\beta\theta = \mathbf{1}_Q$. $\square$

A split exact sequence remains split after any base change. In particular, it remains split after localization. There are partial converses. Recall that if $I \subseteq R$,

$$\mathcal{V}(I) = \{P \in \mathrm{Spec}\,(R) : I \subseteq P\},$$

and that

$$\mathcal{D}(I) = \mathrm{Spec}\,(R) - \mathcal{V}(I).$$

In particular,

$$\mathcal{D}(fR) = \{P \in \mathrm{Spec}\,(R) : f \notin P\},$$

and we also write $\mathcal{D}(f)$ or $\mathcal{D}_f$ for $\mathcal{D}(fR)$.

Theorem. *Let R be an arbitrary ring and let*

$$(\#) \quad 0 \to N \xrightarrow{\alpha} M \xrightarrow{\beta} Q \to 0$$

be a short exact sequence of R-modules such that Q is finitely presented.

(a) *$(\#)$ is split if and only if for every maximal ideal m of R, the sequence*

$$0 \to N_m \to M_m \to Q_m \to 0$$

is split.

(b) *Let S be a faithfully flat R-algebra. The sequence (#) is split if and only if the sequence*

$$0 \to S \otimes_R N \to S \otimes_R M \to S \otimes_R Q \to 0$$

is split.

(c) *Let W be a multiplicative system in R. If the sequence*

$$0 \to W^{-1}N \to W^{-1}M \to W^{-1}Q \to 0$$

is split over $W^{-1}R$, then there exists a single element $c \in W$ such that

$$0 \to N_c \to M_c \to Q_c \to 0$$

is split over R_c.

(d) *If P is a prime ideal of R such that*

$$0 \to N_P \to M_P \to Q_P \to 0$$

is split, there exists an element $c \in R - P$ such that

$$0 \to N_c \to M_c \to Q_c \to 0$$

is split over R_c. Hence, (#) becomes split after localization at any prime P' that does not contain c, i.e., any prime P' such that $c \notin P'$.

(e) *The split locus for (#), by which we mean the set of primes $P \in \mathrm{Spec}\,(R)$ such that*

$$0 \to N_P \to M_P \to Q_P \to 0$$

is split over R_P, is a Zariski open set in $\mathrm{Spec}\,(R)$.

Proof. Let $C = \mathrm{Coker}\,\big(\mathrm{Hom}(Q,\, M) \to \mathrm{Hom}_R(Q,\, Q)\big)$, as in the preceding Proposition, and let γ denote the image of $\mathbf{1}_Q$ in C. By part (4) of the preceding Propostion, (#) is split if and only if $\gamma = 0$.

(a) The "only if" part is clear, since splitting is preserved by any base change. For the "if" part, suppose that $\gamma \neq 0$. The we can choose a maximal ideal m in the support of $R\gamma \subseteq C$, i.e., such that $\mathrm{Ann}_R\gamma \subseteq m$. The fact that Q is finitely presented implies that localization commutes with Hom. Thus, localizing at m yields

$$0 \to \mathrm{Hom}_{R_m}(Q_m,\, N_m) \to \mathrm{Hom}_{R_m}(Q_m,\, M_m) \to \mathrm{Hom}_{R_m}(Q_m,\, Q_m) \to C_m \to 0,$$

and since the image of γ is not 0, the sequence $0 \to N_m \to M_m \to Q_m \to 0$ does not split.

(b) Again, the "only if" part is clear, and since Q is finitely presented and S is flat, Hom commutes with base change to S. After base change, the new cokernel is $S \otimes_R C$. But $C = 0$ if and only if $S \otimes_R C = 0$, since S is faithfully flat, and the result follows.

(c) Similarly, the sequence is split after localization at W if and only if the image of γ is 0 after localization at W, and this happens if and only if $c\gamma = 0$ for some $c \in W$. But then localiziing at the element c kills γ.

(d) This is simply part (c) applied with $W = R - P$

(e) If P is in the split locus and $c \notin P$ is chosen as in part (d), $\mathcal{D}(c)$ is a Zariski open neighborhood of P in the split locus. $\square$

Flat extensions of Cohen-Macaulay rings

Recall that if P is a prime ideal of R and we have a ring homomorphism $h : R \to S$, then with $\kappa_P = R_P/PR_P \cong \mathrm{frac}\,(R/P)$, then $\kappa_P \otimes_R S \cong (R-P)^{-1})(S/PS)$ is called the *fiber* over P. In fact, $\mathrm{Spec}\,(\kappa_P \otimes_R S)$ is homeomorphic with subspace of $\mathrm{Spec}\,(S)$ consisting of primes that lie over P, i.e., with $(\mathrm{Spec}\,(h))^{1-1}(P)$. If $R = (R, m)$ is lcoal, the fiber over m is called the *closed fiber) of $R \to S$. We want to show the following:*

Theorem. *Let $R \to S$ be a flat homomorphism of Noetherian rings. If R is Cohen-Macaulay and all the fibers $\kappa_P \otimes_R S$ are Cohen-Macaulay for P in* $\mathrm{Spec}\,(R)$*, then R is Cohen-Macaulay.*

The key to proving this result is the following:

Theorem. *Let $(R, P) \to (S, Q)$ be a flat local homomorphism (so that P maps into Q). Then:*

(a) $\dim\,(S) = \dim\,(R) + \dim\,(S/PS)$.

(b) $\mathrm{depth}(S) = \mathrm{depth}(R) + \mathrm{depth}(S/PS)$.

(c) S is Cohen-Macaulay if and only if both R and S/PS are Cohen-Macaulay.

(d) If $y_1, \ldots, y_k \in Q$ is a regular sequence on S/PS, then $y_1, \ldots, y_k$ is a regular sequence on $M \otimes S$ for every nonzero R-module M, and $S/(y_1, \ldots, y_k)$ is again R-flat. In particular, $y_1, \ldots, y_k$ is a regular sequence on S.

Before giving the proof, note that this implies the Theorem stated first: it suffices to show that S_Q is Cohen-Macaulay for all primes Q. But if Q lies over P, we have that $R_P \to S_Q$ is flat local, and both R_P and S_Q/PR_P are Cohen-Macaulay, since the latter is a localization of the fiber $(R-P)^{-1}S/PS$.

Proof. Throughout, note that for every ideal $I \subseteq P$ of R, $R/I \to S/IS$ is again flat, by base change, and the closed fiber does not change.

(a) We use induction $\dim\,(R)$. Let N be the ideal of nilpotents in R. Then NS also consists of nilpotents. We may replace R by R/NR and S by S/NS. The dimensions don't change. Thus, we may assume that R is reduced. If $\dim(R) = 0$, then R is a field, $P = 0$, $S/PS \cong S$, and the result is clear. If $\dim(R) \geq 1$, we can choose $x \in P$ not in any minimal prime. Since R is reduced, all associated primes are 0, and x is a nonzerodivisor in R and, hence, in S. By the induction hypothesis, $\dim(S/xS) = \dim(R/xR) + \dim(S/PS)$, and we have that $\dim(S) = \dim(S/xS) + 1$ and $\dim(R) = \dim(R/xR) + 1$.

(d) By iinduction of k this reduces at once to the case were $k = 1$, and we write $y = y_1$. We first prove that y is a nonzerodivisor on S/IS for every proper ideal I of R (including (0). Let $\mathcal{I}$ be the set of ideals I such that y is a zerodivisor on S/IS. If this set is nonempty, choose a maximal element I_0 and replace $R \to S$ by $R/I_0 \to S/I_0$. Thus, we may assume without loss of generality that if I is any nonzero ideal of R contained in P, then y is a nonzerodivisor on S/IS, but that y is a zerodivisor on S. If P contains a nonzero divisor x, it follows that x, y is a regular sequence on S (y is a nonzerodivisor on S/xS). Since regular sequences are permutable in the local case, y is a nonzerodivisor on S, a contradiction. Hence, P is an associated prime of R, and we can choose $u \in R$ with annihilator P. If $u \not{in} P$ then u is a unit, P kills R, and so $P = (0)$ and there is nothing to prove. If $u \in P$, consider the exact sequence $0 \to R/P \to R \to R/uR \to 0$, where R/P is the submodule of R generated by u. We may tensor with S over R to obtain an exact sequence $0 \to S/PS \to S \to S/uS \to 0$. By hypothesis, y is a nonzerodivisor on S/PS, and by the hypothesis of Noetherian induction, y is a nonzerodivisor on S/uS. It follows that y is a nonzerodivisor on S after all. We next need to show that y is a nonzerodivisor on $M \otimes_R S$ for every nonzero R-module M. Since M is a directed union of finitely generated R-modules, we may assume that M is finitely generated. Then M has a finite filtration with factors R/I_j, where the I_j are ideals of R, and $S \otimes_R M$ has a finite filtration by modules S/I_jS. By what we have already shown, y is a nonzerodivisor on each factor, and so it is a nonzerodivisor on $S \otimes_R M$. Finally, we must show that S/yS is again R-flat. But if $0 \to M_1 \to M_2 \to M_3 \to 0$ is an exact sequence of R-modules, then $0 \to S \otimes_R M_1 \to S \otimes_R \to M_2 S \otimes_R M_3 \to 0$ is an exact sequence of S-modules and y is not a zerodivisor on each of these. It follows that tensoring with S/yS over S preserves exactness, and so tensoring the original sequence with S/yS over R preserves exactness.

(b) We choose a maximal regular sequence $x_1, \ldots, x_d$ in P on R and a maximal regular sequence $y_1, \ldots, y_{d'}$ in Q on S/PS. Then $x_1, \ldots, x_d$ is a regular sequence in S and we may replace $R \to S$ by $R/(x_1, \ldots, x_d)R \to S/(x_1, \ldots, x_d)S$ and assume that R has depth 0. Then $y_1, \ldots, y_{d'}$ is a regular sequence on S and we may replace S by $S/(y_1, \ldots, y_{d'})S$. Thus, it will suffice to show that if R and S/PS both have depth 0, then so does S. Choose an embedding $R/P \hookrightarrow R$. This yields an embedding $S/PS \hookrightarrow S$, and since S/PS has an element killed by Q, so does S.

(c) follows from (a) and (b) and the fact that the depth of a local ring is always at most its dimension.

□

Corollary. *A polynomial ring in a finite number of variables over a Cohen-Macaulay ring is Cohen-Macaulay.*

Proof. This reduces to the case of one variable. The fibers all have the form $\kappa_P[x]$, and one-dimensional domains are Cohen-Macaulay. $\square$

Corollary. *Cohen-Macaulay rings are universally catenary (and hence so are homomorphic images of Cohen-Macaulay rings.* $\square$