Math 615, Winter 2016 **Problem Set #1 Solutions**

1. For positive integers a, b, $x_1^a >_\gamma x_2^b \Leftrightarrow a\gamma > b \Leftrightarrow \gamma > b/a$ from which we have $\gamma = \sup\{b/a : x_1^a >_\gamma x_2^b\}$. It follows that if $>_\gamma$ and $>_{\gamma'}$ are the same order then $\gamma = \gamma'$. $\square$

2. (a) $\mu \neq 1 \Rightarrow W(\mu) > 0 \Rightarrow$ for all $k \gg 0$ $W(\mu^k) = kW(\mu) > W(\mu')$, while $x_1 >_{\text{lex}} x_2^k$ for all k.

(b) Choose integers a, $b > 0$ such that $\gamma_2/\gamma_1 < a/b < 1$, so that $a\gamma_1 > b\gamma_2$ but $a < b$. Then $\deg(x_1^a) < \deg(x_2^b)$ but $x_1^a > x_2^b$, showing that $<$ is neither hlex nor revlex.

3. In all three parts, we know that the standard relations on pairs of elements generate the module of relations. In each case we must omit terms as needed so that these relations become minimal.

(a) The standard relations $(0,\, x_3^5,\, -x_2^3)$, $(x_3^5,\, 0,\, -x_1^2)$, and $(x_2^3, -x_1^2,\, 0)$ are already minimal, since none of x_1^2, x_2^3, x_3^5 is a multiple of either of the others (if one of the three were a sum of multiples of the other two, one of its nonzero entries would be forced to be such a multiple).

(b) The GCD is $x_1^{11}x_2^8$, and so there is just one standard relation $(x_2^4,\, -x_1^8)$, which must be minimal.

(c) Let $\mu_i = x_i x_{i+1}$, $1 \le i \le n-1$. The standard relations θ_{ij} on μ_i, μ_j, where $1 \le i < j \le n-1$ and $j \neq i+2$ give a minimal set of relations. For the case $j = i+1$, $1 \le i \le n-1$, the relation on $x_i x_{i+1}$ and $x_{i+1}x_{i+2}$, takes the form $x_{i+2}e_i - x_i e_{i+1}$. When $j > i+1$, the relation takes the form $x_j x_{j+1} e_i - x_i x_{i+1} e_j$. The relations $\theta_{i,i+2}$ are not needed, since $\theta_{i,i+2} = x_{i+2}x_{i+3}e_i - x_i x_{i+1} e_{i+2} = x_{i+3}(x_{i+2}e_i - x_i e_{i+1}) + x_i(x_{i+3}e_{i+1} - x_{i+1}e_{i+3}) = x_{i+3}\theta_{i,i+1} + x_i\theta_{i+1,i+2}$. None of remaining θ_{ij} is an R-linear combination of the others because the coefficient on e_i in this relation is not in the ideal generated by the coefficients of e_i in the other θ_{ik}, $k > i$, $k \neq j$, $k \neq i+2$ (for $k > i$, $k \neq i+2$, we get $x_{i+1}, x_k x_{k+1}$, $k \geq i+3$) or other θ_{hi}, $h < i$, $h \neq i-2$ (for these, we get $-x_i$ and $-x_h x_{h+1}$, $h < i-1$).

4. Any element f in I is a sum of polynomials of distinct degrees, say $g_1, \ldots, g_h$, each of which is in I. The terms of f are the elements in the union of the sets of terms of the various g_i. Hence, the initial term of f must the largest of the initial terms of the g_i, and this is the initial term of a homogeneous element of I.

5. (a) $G_{1,2} = x_2 g_1 - x_1 g_2 = -x_1^2 x_3 + x_2 x_3^2$. Since neither term is divisible by $\text{in}(g_1)$ nor $\text{in}(g_2)$, this is already the remainder in a standard expression for division by g_1, g_2, and we let $g_3 = -x_1^2 x_3 + x_2 x_3^2$. Then $G_{1,3} = x_1 x_3 g_1 + x_2 g_3 = x_1 x_3^3 + x_2^2 x_3^2 = x_3^2 g_2 + 0$, while $G_{2,3}$ need not be checked since the initial terms are relatively prime. Hence, g_1, g_2, g_3 is a Gröbner basis.

(b) Since the initial terms are relatively prime, h_1, h_2, is already a Gröbner basis.

6. For $k = i$, j, let $g_k = c_k \mu_k e_t + \widetilde{g}_k e_t$, where $\widetilde{g}_k$ is a polynomial such that all terms of $\widetilde{g}_k e_t$ are smaller than $c_k \mu_k e_t$. Since all terms involve e_t we omit it from the notation. We have that $\text{GCD}(\mu_i,\, \mu_j) = 1$. Then $G_{ij} = c_j \mu_j g_i - c_i \mu_i g_j = (g_j - \widetilde{g}_j)g_i - (g_i - \widetilde{g}_i)g_j = -\widetilde{g}_j g_i + \widetilde{g}_i g_j + 0$. This will be the required standard expression. If either or both of $\widetilde{g}_i$ or

$\widetilde{g_j}$ is 0 this is clear. If not, it suffices to show that the initial terms of the two summands do not cancel, for then one is the initial term of G_{ij} and the other is no larger. But if they cancel then $\mathrm{in}(\widetilde{g_j})\mu_i$ and $\mathrm{in}(\widetilde{g_i})\mu_j$ are the same up to scalar multiplication, and since $\mathrm{GCD}(\mu_I,\,\mu_J)=1$, we must have that $\mu_i|\mathrm{in}(\widetilde{g_i})$, a contradiction, since $\mathrm{in}(\widetilde{g_i})<\mu_i$. $\square$

7. The initial forms of the given elements are:
$x_1x_2,\ x_3x_4x_5,\ x_6x_7x_8x_9,\ \dots\ ,\ x_{\binom{k+1}{2}}x_{\binom{k+1}{2}+1}\cdots x_{\binom{k+1}{2}+k},\ \dots\ ,\ x_{\binom{n+1}{2}}x_{\binom{n+1}{2}+1}\cdots x_{\binom{n+1}{2}+n}.$
These are relatively prime in pairs: hence, one need not perform any tests in the Buchberger algorithm, and the given elements are a Gröbner basis for the ideal they generate. The displayed initial forms span $\mathrm{in}(I)$. No initial form divides any other, so this is a minimal Gröbner basis. No initial term divides any term of any other element of the Gröbner basis, the coefficients in the initial terms are all 1, and the initial terms are decreasing. Hence, the given elements are already a reduced Gröbner basis for the ideal.

8. For $1\le i<j\le n$, let $D_{i,j}$ denote $x_ix_{n+j}-x_jx_{n+i}$, the 2×2 minor formed from the columns of x_i and x_j. We claim that $D_{1,2},\ \dots D_{1,n},\ D_{2,3},\ \dots\ ,\ D_{2,n},\ \dots\ ,\ D_{n-1,n}$ is already a reduced Gröbner basis. Since the initial forms x_ix_{n+j} for $i<j$ are already decreasing with coefficient 1 and no term divides any other, we need only apply the (sharpened) Buchberger criterion. The only cases where the initial terms are *not* relatively prime and need to be checked are (1) $D_{i,j},\ D_{i,k}$ and (2) $D_{i,k},\ D_{j,k}$, where $i<j<k$ in both cases. In case (1), we have $x_{n+k}D_{ij}-x_{n+j}D_{ik}=-x_{n+k}x_jx_{n+i}+x_{n+j}x_kx_{n+i}=-x_{n+i}D_{j,k}+0$, which is already a standard expression for division by the proposed Gröbner basis with remainder 0, since there is only one term. In case (2), similarly, we have that $x_jD_{i,k}-x_iD_{j,k}=-x_jx_kx_{n+i}+x_ix_kx_{n+j}=x_kD_{i,j}+0$. $\square$