Math 615, Winter 2016 **Problem Set #2**
Due: Friday, February 26

Throughout, K is a field, and $K \subseteq L$ is a field extension.

1. Let $K[x_1, \ldots, x_n]$ be the polynomial ring in n variables over K. Prove that if $\mathrm{in}(I)$ is generated by square-free monomials, i.e., is a radical ideal, then I is a radical ideal. (It follows that the ideal in Problems 7. and 8. of the first problem set is radical.)

2. Let Σ be a finite simplicial complex with vertices $x_1, \ldots, x_n$ and let I_Σ be the ideal of the polynomial ring $R = K[x_1, \ldots, x_n]$ generated by the square-free monomials that are products of a set of variables that is *not* a face of Σ. Prove the fact stated in class that the minimal primes of R/I_Σ are in bijective correspondence with the facets (maximal faces) of Σ, where the prime corresponding to $\sigma \in \Sigma$ is generated by the image of $\{x_1, \ldots, x_n\} - \sigma$.

3. (a) R, S are finitely generated $\mathbb{N}$-graded K-algebras with $R_0 = S_0 = K$, and $T = R \otimes_K S$ is $\mathbb{N}$-graded by $[R \otimes_K S]_d = \bigoplus_{i+j=d} R_i \otimes_K S_j$. Prove: $\mathfrak{P}_T(z) = \mathfrak{P}_R(z)\mathfrak{P}_S(z)$.

(b) Let $R = K[x_1, \ldots, x_n]$ be the polynomial ring in n variables over K, but graded so that $\deg(x_i) = m_i$ for $1 \leq i \leq n$. (Note: when R is not necessarily generated by one-forms, the Hilbert-Poincaré series and the Hilbert function are defined, but the Hilbert function *need not* be eventually polynomial.) Prove that $\mathfrak{P}(R) = \prod_{i=1}^{n} 1/(1 - z^{m_i})$.

4. Recall that an R-module or algebra S is *flat* over R if $S \otimes_R \_$ preserves exactness (preserving injectivity of maps of modules suffices), and *faithfully flat* over R if it is R-flat and $M \neq 0 \Rightarrow S \otimes_R M \neq 0$ (this implies a sequence of R-modules is exact iff its tensor product with S is exact). Nonzero free modules are faithfully flat, and if $K \subseteq L$ are fields, L is faithfully flat over K, since it is free. Prove the following statements:

(a) If $x \in R$ is a nonzerodivisor on M and S is R-flat then x is a nonzerodivisor on $S \otimes_R M$.

(b) If $x_1, \ldots, x_n \in R$, M is an R-module, and S is faithfully flat over R, then $x_1, \ldots, x_n$ is a regular sequence on M iff it is a regular sequence on $S \otimes_R M$.

(c) If R is a finitely generated K-algebra and $\dim(R) = n$, so that by the Noether Normalization Theorem, R is a module-finite over a polynomial subring $K[x_1, \ldots, x_n] \subseteq R$, then $L[x_1, \ldots, x_n] \subseteq L \otimes_K R$ is module-finite, and so $\dim(L \otimes_K R) = n$.

(d) If R is finitely generated and $\mathbb{N}$-graded over $R_0 = K$ and $F_1, \ldots, F_n$ is a homogeneous system of parameters, then $1 \otimes F_1, \ldots, 1 \otimes F_n \in L \otimes_K R$ is a homogeneous system of parameters; moreover, $L \otimes_K R$ is Cohen-Macaulay iff R is Cohen-Macaulay.

5. Let $R = K[x_1, x_2, x_3, x_4]$ and let $I = (x_1, x_2) \cap (x_3, x_4)$. Find a homogeneous system of parameters for R/I, and verify that it is not a regular sequence in R/I.

6. Let I and J be monomial ideals in the polynomial rings $R = K[x_1, \ldots, x_n]$ and $S = K[y_1, \ldots, y_r]$ respectively such that the respective Hilbert-Poincaré series of R/I and S/J are $F(z)$ and $G(z)$. Let $T = K[x_1, \ldots, x_n, y_1, \ldots, y_r]$, the polynomial ring in $n + r$ variables over K. Prove that the Hilbert-Poincaré series of T/IJT is given by $F(z)/(1-z)^r + G(z)/(1-z)^n - F(z)G(z)$. (It may be helpful that $IJT = IT \cap JT$: if you use this, prove that it is true.)

7. Let K be algebraically closed, let X be an $n \times n$ matrix of indeterminates (x_{ij}) over K, and let $R = K[x_{ij} : 1 \leq i, j \leq n]$. Let S_i denote the set of symmetrically placed minors of X of size i (those minors such that the set of indices of the rows is equal to the set of indices of the columns), and let F_i denote a linear combination of the elements of S_i with coefficients in $K - \{0\}$.

(a) Show that $F_1, \ldots, F_n$ is a regular sequence, and that $I = (F_1, \ldots, F_n)R$ is a radical ideal. [Suggestion: revlex for a suitable ordering of the variables may be helpful.]

(b) Consider the case where for every i, $1 \leq i \leq n$, F_i is the sum of the elements of S_i. You may assume that the elements $(-1)^i F_i$ are the coefficients of the characteristic polynomial of X. Prove that I is prime in this case by showing that $V(I)$ is irreducible. [Suggestion: show that there is a surjection $\mathrm{GL}(n, K) \times M \to V(I)$ where M consists of upper triangular matrices with main diagonal zero and $(A, B) \mapsto ABA^{-1}$.]

8. Let $n \geq 2$ be an integer, let $R = K[x_1, \ldots, x_n]$ and let I_n be generated by all products $x_i x_j$ such that i and j are distinct and are not consecutive integers. What are the minimal primes of R/I, and what is its Krull dimension? What is a homogeneous system of paramters for R/I? Is R/I Cohen-Macaulay? Prove your answers.