Math 615, Winter 2016 **Problem Set #2 Solutions**

1. For any monomial order on R, when $f,\ g \in R - \{0\}$, $\mathrm{in}(fg) = \mathrm{in}(f)\mathrm{in}(g) \Rightarrow \mathrm{in}(f^k) = \big(\mathrm{in}(f)\big)^k$. If $f \in \mathrm{Rad}\,(I)$ then $f^k \in I$ for some k, and $\big(\mathrm{in}(f)\big)^k = \mathrm{in}(f^k) \in \mathrm{in}(I)$, which is radical, and so $\mathrm{in}(f) \in \mathrm{in}(I)$. Hence $\mathrm{in}(I) \subseteq \mathrm{in}\big(\mathrm{Rad}\,(I)\big) \subseteq \mathrm{in}(I) \Rightarrow \mathrm{in}(I) = \mathrm{in}\big(\mathrm{Rad}\,(I)\big)$ and so $\mathrm{Rad}\,(I) = I$. $\square$

2. $I_\Sigma \subseteq P \in \mathrm{Spec}\,(R) \Rightarrow P$ meets each non-face $\tau \subseteq \{x_1,\,\dots\,,x_n\}$: $\prod_{x_i\in\tau} x_i \in I_\Sigma \subseteq P$. Conversely, if $\tau \subseteq \{x_1,\,\dots\,,x_n\}$ meets every non-face, τ generates a prime $\supseteq I_\Sigma$. Hence, minimal primes of I_Σ correspond to and are generated by minimal sets $\tau \subseteq \{x_1,\,\dots\,,x_n\}$ that meet every non-face of Σ. If $\tau \subseteq \{x_1,\,\dots\,,x_n\}$ generates a prime, $\sigma = \{x_1,\,\dots\,,x_n\} - \tau$ must be a face: the product of the x_i in it is $\neq 0$ even in the quotient. Also, if $\sigma \in \Sigma$, then $\sigma' = \{x_1,\,\dots\,,x_n\} - \sigma$ must meet every non-face τ: $\tau \cap \sigma' = \emptyset \Rightarrow \tau \subseteq \sigma$. It follows at once that the minimal primes of I_Σ correspond bijectively with the maximal faces of Σ. $\square$

3. (a) Let $\mathfrak{P}_R(z) = \sum_{i=0}^\infty a_i z_i$ and $\mathfrak{P}_S(z) = \sum_{j=0}^\infty b_j z^j$ By the definition of multiplication for power series, the coefficient of z^n in the product is $\sum_{i+j=n}^n a_i b_j$, where $i,\ j \in \mathbb{N}$, and since $\dim_K(R_i \otimes_K S_j) = a_i b_j$, the result follows. Since $K[x_1,\,\dots\,,x_n]$ is the iterated tensor product of the rings $R_i = K[x_i]$, and $\mathfrak{P}_{R_i}(z) = \sum_{t=0}^\infty (z^{m_i})^t = 1/(1-z^{m_i})$, (b) follows. $\square$

4. (a) Consider $0 \to M \xrightarrow{x\cdot} M$. If M is flat (respectively, faithully flat), it remains exact (respectively, its exactness is not affected) after applying $S \otimes_R \_$. Then (b) follows: by (a), each x_{i+1} is a nonzerodivisor on $M_i = M/(x_1,\,\dots\,,x_i)M$ iff it is a nonzerodivisor on $W_i = S \otimes_R M_i \cong (S \otimes_R M)/(x_1,\,\dots\,,x_i)(S \otimes_R M)$; moreover, $M_n \neq 0 \iff W_n \neq 0$ by *faithful* flatness. $\square$ (c) is clear, since $L \otimes_K \_$ preserves module-finiteness (the same generators work), and module-finite (or even integral) extensions preserve dimension. (d) is immediate from (c), the fact that $F_1,\,\dots\,,F_n$ is a homogeneous system of parameters iff R is module-finite over the polynomial ring $K[F_1,\,\dots\,,F_n]$, the faithful flatness of L over K, the definition of the Cohen-Macaulay property, and part (b).

5. The quotient of $R/(x_1,x_2) \cap (x_3,x_4) = R/(x_1x_3, x_1x_4, x_2x_3, x_2x_4)$ by each minimal prime has dimension 2, and so $\dim\,(R/I) = 2$. Mod $(x_1 - x_3,\ x_2 - x_4)$, we obtain $K[x_1,x_2]/(x_1^2, x_1x_2, , x_2^2)$, which is 0-dimensional. Hence, $x_1 - x_3$, $x_2 - x_4$ is a homgeneous system of parameters. Mod $(x_1 - x_3)$ R/I becomes $K[x_1,\ x_2,\ x_4]/(x_1^2, x_1x_4, x_1x_2, x_2x_4)$, The image of $x_2 - x_4$ is a zerodivisor, since it kills the image of x_1, which is nonzero because it has too small a degree to be in $(x_1^2,\ x_1x_2,\ x_2^2)$. $\square$

6. Since intersection distributes over sum for monomial ideals, $IT \cap JT$ is the sum of the ideals $\mu T \cap \nu T$ where μ and ν are monomial generators of I and J, respectively. Since μ and ν involve disjoint sets of variables, every $\mu T \cap \nu T = \mu\nu T$. Hence, $IT \cap JT = IJT$. We have a short exact sequence $(*)$ $\quad 0 \to T/\big((IT) \cap (JT)\big) \to T/IT \bigoplus T/JT \to T/(I+J)T \to 0$, where the maps preserve degree. The leftmost nonzero term is T/IJT. Now $T/IT \cong (R/I) \otimes_K S$ and so, by Problem 3., $\mathfrak{P}(T/IT) = \dfrac{F(z)}{(1-z)^r}$. Similarly, $\mathfrak{P}(T/JT) = \dfrac{G(z)}{(1-z)^n}$. Also, $T/(I+J)T \cong (R/I) \otimes (S/J)$ and so $\mathfrak{P}_{T/(I+J)T}(z) = F(z)G(z)$. The alternating sum of the Hilbert-Poincaré series of the terms in the exact sequence $(*)$ is zero. $\square$

7. (a) Order the variables in the matrix such that for each positive integer $h \geq 2$, the variables x_{ij} such that $i+j = h+1$ are smaller than the variables such $i+j = h$. We refer to the variables x_{ij} such that $h+1$ as being on the hth *antidiagonal.* The expansion of the $k \times k$ minor in the upper left corner contains a term that is, up to sign, the product G_k of the variables on the kth antidiagonal. Any other product occurring in the expansion of any other symmetrically placed $k \times k$ minor contains a variable x_{ij} such that $i+j > k$, which will be smaller in revlex. Hence, given that the coefficients of the symmetrically placed minors are all nonzero, the initial term of the sum F_k is, up to multiplication by a scalar, G_k. Since the G_k are relatively prime in pairs, it follows from the Buchberger algorithm that $G_1, \ldots, G_k$ generate the initial ideal of $I_k := (F_1, \ldots, F_k)$ for $1 \leq h \leq n$. Since the G_k are square-free, they generate a radical ideal, and, hence, so do the F_k. Since G_{k+1} is a product of variables not occurring in Gh for $h \leq k$, $G_1, \ldots, G_n$ is a regular sequence. To see that F_{k+1} is a nonzerodivisor mod I_k, suppose that $HF_{k+1} \in (F_1, \ldots, F_k) = I_k$. Divide H by $F_1, \ldots, F_k$. This will give a remainder H_0 with no term in $(G_1, \ldots, G_k)$. But $\mathrm{in}(H_0F_{k+1}) = \mathrm{in}(H_0)G_{k+1} \in (G_1, \ldots, G_k)$, and so $\mathrm{in}(H_0) \in (G_1, \ldots, G_k)$, a contradiction unless $H_0 = 0$. $\square$

(b) Since the ideal is known to be radical, it suffices to show that the algebraic set it defines is irreducible, The coefficients of the characteristic polynomial of an $n \times n$ matrix vanish iff the matrix is nilpotent. An upper triangular matrix with all entries 0 on the main diagonal is nilpotent, and by using Jordan form, a matrix is nilpotent iff it is similar to a an upper triangular matrix with zero entries on the main diagonal. That is, $V(I)$ is the image of the map described in the statement of the problem. Since $\mathrm{GL}(n, K)$ is a nonempty open set in $\mathbb{A}^{n^2}$, it is a variety, while $M \cong \mathbb{A}^{n(n-1)/2}$. Since the product of two varieties over an algebraically closed field is a variety, and the image of an irreducible set under a continuous map is irreducible, it follows that $V(I)$ is irreducible. $\square$

8. I corresponds to the simplicial complex whose edges are

$$\{x_1, x_2\}, \ldots, \{x_i, x_{i+1}\}, \ldots, \{x_{n-1}, x_n\}.$$

This simplicial complex corresponds to a closed line segment L. It's dimension is one, and so the Krull dimension of the ring is 2. The minimal primes are given by the complements of the maximal faces: there are $n-1$ of these, $P_1, \ldots, P_{n-1}$, where P_i is generated by all of the variables except $\{x_i, x_{i+1}\}$. Note that the quotient by any minimal prime P_i is $\cong K[x_i, x_{i+1}]$, which again proves that the dimension of the quotient is 2. Let f be the sum of the x_i for i even and let g be the sum of the x_i for i odd. As we multiply f (resp., g) by the even (resp., the odd) numbered variables we obtain all the squares of the even (resp., odd) numbered variables. Therefore, m is nilpotent mod (I_n, f, g) and f, g is a homogeneous system of parameters for $S_n := R/I_n$. The Cohen-Macaulay property for this ring follows immediately from Reisner's criterion: the reduced simplicial cohomology of L with coefficients in any field is 0, and the links are 0-dimensional, so there is no condition (the top dimension for each link is 0). However, we give a direct proof by induction on n that f, g is a regular sequence. Note that when $n = 2$ the ring is simply $K[x_1, x_2]$. Here, $f = x_2$ and $g = x_1$. (When $n = 2$, the ring is $K[x_1, x_2, x_3]/(x_1x_3)$ and it is correct that $x_1 + x_3$ and x_2 is a regular sequence, but the inductive argument below makes it unnecessary to check this case separately.) Let $J = P_2 \cap \cdots \cap P_n$, so that $I = P_1 \cap J$. Then we have an exact sequence

$$(*) \quad 0 \to R/I_n \to R/P_1 \oplus R/J \to R/(P_1 + J) \to 0.$$

All of $P_2, \ldots, P_n$ contain x_1. Since all of these are monomial ideals, intersection distributes over sum, and $J = (x_1) + J_1$ where J_1 is the intersection of the primes $Q_2, \cdots, Q_{n-1}$ generated by the variables in $\{x_2, \ldots, x_n\}$ with x_i and x_{i+1} omitted for $2 \leq i \leq n-1$. Let q_i be the prime of $K[x_2, \ldots, x_n]$ generated by all the variables except x_i and x_{i+1}, $2 \leq i \leq n$. Then $R/J \cong K[x_2, \ldots, x_n]/J_0$, where J_0 is the intersection of $q_2, \ldots, q_n$. Then $R/J \cong K[x_1, \ldots, x_{n-1}]/I_{n-1} = S_{n-1}$ by shifting the numbering of the variables by 1, and the images of f and g give the corresponding elements, up to order, in S_{n-1}. Hence, f, g is a regular sequence on R/J, and this is also true for $R/P_1 \cong K[x_1, x_2]$. Note also that f is a nonzerodivisor on $R/(P_1 + J)$, since the denominator ideal is the same as the intersection of the $P_1 + P_j$ for $j \geq 2$, and $P_1 + P_2$ is generated by all variables except x_2 while the $P_1 + P_j$ for $j > 2$ are generated by all the x_i. Thus, the denominator $P_1 + J$ is the same as $P_1 + P_2$, and the quotient is $K[x_2]$. Thus, f is a nonzerodivisor on all three modules, so that the sequence $(*)$ remains exact when we tensor with R/fR. This yields an injection $0 \to R/(I_n + fR) \to R/(P_1 + fR) \oplus R/(J + fR)$. Since f, g is a regular sequence on each of R/P_1 and R/J, it follows that g is a nonzerodivisor on $R/(P_1 + fR) \oplus R/(J + fR)$, and, hence, on its submodule $R/(I_n + fR)$. $\square$