Math 615, Winter 2016 **Problem Set #3**
Due: Wednesday, March 16

1. Let (R, m, K) be a local ring, and let $\widehat{R}$ be its m-adic completion. Show that R is Cohen-Macaulay if and only if $\widehat{R}$ is Cohen-Macaulay.

2. Let R be a Cohen-Macaulay ring. Show that the formal power series ring $R[[x_1, \ldots, x_n]]$ is Cohen-Macaulay. (Note that when $n = 1$, with $x = x_1$, every maximal ideal of $R[[x]]$ contains x, and so has the form $mR[[x]] + xR[[x]]$, where m is a maximal ideal of R.)

3. Let R be a Noetherian ring and let I be an ideal of R. Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of finitely generated modules for each of which depth on I is finite (i.e., $IM' \neq M'$, $IM \neq M$, and $IM'' \neq M''$).

(a) Prove that $\mathrm{depth}_I M \geq \min\{\mathrm{depth}_I M', \mathrm{depth}_I M''\}$.

(b) Prove that if $\mathrm{depth}_I M > \mathrm{depth}_I M''$, then $\mathrm{depth}_I M' = \mathrm{depth}_I M'' + 1$.

4. Let $R = K[x_1, \ldots, x_n]$ be a polynomial ring with homogeneous maximal ideal $m = (x_1, \ldots, x_n)R$, and let M be a finitely generated nonzero $\mathbb{Z}$-graded module. Recall the class result that M is free if and only if $\mathrm{depth}_I M = n$. Show that if $\mathrm{depth}_m M = n - k < d$, then any graded kth module of syzygies of M is free, while if $0 \leq h < k$, no hth module of syzygies of M is free. (This is the graded form of the Hilbert Syzygy Theorem.)

5. A group G acts by ring automorphisms on a ring R. If R is a domain, the action of G extends uniquely to an action on the fraction field $\mathcal{F}$ of R (you need not prove this.)

(a) Prove that if G is finite and R is a domain, then $\mathrm{frac}\,(R^G) = \mathcal{F}^G$.

(b) Let $G = K - \{0\}$ under $\cdot$ act on the polynomial ring $R = K[x, y]$ over the field K by letting $a : x \mapsto ax$ and $a : y \mapsto ay$. Is it true that $\mathrm{frac}\,(R^G) = \mathcal{F}^G$ in this case?

(c) Suppose that the domain R is integrally closed in $\mathcal{F}$. Prove that R^G is integrally closed in its fraction field.

(d) Suppose that G is a finite group such that the order $h = |G|$ of G is invertible in R. Show that $R^G \subseteq R$ splits as a map of R^G-modules.

(e) Show that if $R \subseteq S$ is a module-finite extension of Noetherian rings such that the inclusion map splits as a map of R-modules and S is Cohen-Macaulay, then R is Cohen-Macaulay. [Hence, under the same hypothesis as in (e), if R is module-finite over R^G, which is automatic if R is finitely generated over a field K and G consists of K-algebra automorphisms) and R is Cohen-Macaulay, then R^G is Cohen-Macaulay.]

6. Let K be a field and let $X = (x_{ij})$ be an $m \times n$ matrix of indeterminates over K, where $2 \leq m \leq n$. Let t be an integer such that $2 \leq t \leq m$. Let Y be an $(m-1) \times (n-1)$ of new indeterminates. Show that $\big(K[X]/I_t(X)\big)_{x_{mn}}$ is isomorphic with a localization of a polynomial ring over $K[Y]/I_{t-1}(Y)$. Hence, if x_{mn} is a nonzerodivisor on $K[X]/I_t(X)$ and $K[Y]/I_{t-1}(Y)$ is an integral domain, then $K[X]/I_t(X)$ is an integral domain.

7. Let notation be as in Problem 6. above with $t = 2$. Find a Gröbner basis for $I_2(X)$ in revlex, with the variables ordered so that $x_{hi} > x_{jk}$ if $h < j$ or $h = j$ and $i < k$. Explain why x_{mn} is not a zerodivisor on $K[X]/I_2(X)$. Prove that $I_2(X)$ is a prime ideal.

8. (a) Let $\mathcal{B}$ be a well-ordered set (i.e., a totally ordered set with DCC). Let $\mathcal{F}$ be the set of finite subsets of $\mathcal{B}$, and put a total ordering on $\mathcal{F}$ as follows: $A > B$ if A contains B strictly or if $A = \{a_1, \ldots, a_k\}$ with $a_1 > \cdots > a_k$, $B = \{b_1, \ldots, b_h\}$ with $b_1 > \cdots > b_h$, and there exists $i \leq \min\{h, k\}$ such that $a_j = b_j$ for $j < i$ while $a_i > b_i$. (E.g., if $a > b > c$ then $\{a, b\} > \{a\}$, $\{a, b\} > \{a, c\}$, and $\{a\} > \{b, c\}$.) Prove that this well-orders $\mathcal{F}$.

(b) Let F be a finitely generated free $K[x_1, \ldots, x_n]$-module with ordered basis. Fix a monomial order on F. One can modify the non-deterministic division algorithm as follows: let $f \in F$ and $g_1, \ldots, g_r \in F$. At each step, if some $\mathrm{in}(g_i)$ divides some term of the "current" remainder f', choose *any* such term ν of f' (it need not be biggest) and subtract a scalar times a monomial multiple of g_i, say $c\mu g_i$, so as to cancel ν (add $c\mu$ to the coefficient of g_i, and subtract $c\mu g_i$ from f'). Prove that this process must terminate after finitely many steps.