Math 615, Winter 2016 **Problem Set #3 Solutions**

1. The map $R \to \widehat{R}$ is a flat local map, and the closed fiber K is Cohen-Macaulay. By a class theorem, $\widehat{R}$ is Cohen-Macaulay if and only if R is Cohen-Macaulay. $\square$

2. By induction on the number of variables, it suffices to prove the result when $n = 1$, i.e., that $R[[x]]$ is Cohen-Macaulay, and it will suffice to show that for every maximal ideal $\mathcal{M}$ of this ring, that $R[[x]]_{\mathcal{M}}$ is Cohen-Macalay. Note that $\mathcal{M}$ contains x, since otherwise $Rx+\mathcal{M}$ is the unit ideal, i.e., contains 1, and we would have have that $1 = rx + u$ with $r \in R$, and $u \in \mathcal{M}$. But $u = 1-rx$ cannot be in cM, since it has inverse $1+rx+r^2x^2+\cdots+r^tx^t+\cdots$. Once we kill x, which is a nonzerodivisor, the dimension and depth both drop by 1, and $\mathcal{M}/xR$ must be a maximal ideal m in $R[[x]]/xR \cong R$. Then $R[[x]]_{\mathcal{M}}/(x) \cong R_m$, which is Cohen-Macaulay, and contains a system of parameters $f_1, \ldots, f_d$ that is a regular sequence. It follows that $x, f_1, \ldots, f_d$ is system of parameters that is a regular sequence in $R[[x]]$. $\square$

3. (a) Identify M' with its image in M. If the depth of M' or M'' is 0, there is nothing to prove. If both are positive, I is not contained in any associated prime of M or M'. Hence, it is not contained in the union of all these associated primes, and we can choose an element x of I that is not a zerodivisor on M' or on M''. Then if x kills an element $m \in M$, it must kill the image of m in M'', which implies that the image of m in M'' is 0. But then $m \in M'$, a contradiction. Thus, x is a nonzerodivisor on all three modules, and the isomorphic sequence of submodules $0 \to xM' \to xM \to xM'' \to 0$ is exact. It follows that $0 \to M'/xM' \to M/xM \to M''/xM'' \to 0$ is exact as well, and the result now follows by induction on $\mathrm{depth}_I M'$. $\square$

(b) If $\mathrm{depth}_I(M'') = 0$ choose $u \in M$ whose image in M'' is killed by I. Then choose any nonzerodivisor $f \in I$ on M. Then f is a nonzerodivisor on $M' \subseteq M$, and $fu \in M'$, since I kills the image of u in $M'' \cong M/M'$. However, $fu \notin fM'$ or else $u \in M'$. But $I(fu) = f(Iu) \subseteq fM'$, so that the image of fu in M'/fM' is a nonzero element killed by I. It follows that f is a maximal regular sequence M'. We complete the proof by induction on $d = \mathrm{depth}_I M''$. If $d > 0$ then all three depths are positive. We can choose $x \in I$ avoiding all three sets of associated primes. It then follows as in part (a) that $0 \to M'/xM' \to M/xM \to M''/xM'' \to 0$ is exact, with all three depths decreased by 1, and the result is immediate by induction on $\mathrm{depth}_I(M')$. $\square$

4. We have $\mathrm{depth}_m R = n$. By Problem 3, each time we take a graded module of syzygies M_1 of M, if $\mathrm{depth}_m M < n$ then $\mathrm{depth}_m M_1 = \mathrm{depth}_m M + 1$. The result is immediate from the given characterization of free modules. $\square$

5. (a) Let $f = r/s$, $r \in R$, $s \in R - \{0\}$, be invariant. Let $t = \prod_{g \in G-\{e\}} g(s)$, so that st is the product of elements in an orbit and is in R^G. Then $f = rt/st$, and since f, st are fixed by G, so is $rt = f(st)$. Hence $rt \in R^G$ and $f \in \mathrm{frac}\,(R^G)$. $\square$

(b) Part (a) takes care of the case where K is finite. If K is infinite, $R^G = K$: a polynomial is fixed by G iff all the monomials in it are fixed by G, and G fixes no monomial except 1. Hence, $\mathrm{frac}\,(R^G) = K$. But $y/x \in \mathcal{F}^G$.

(c) $R^G = R \cap \mathcal{F}^G$, both of which are integrally closed, and so R^G is as well. $\square$

(d) The map $\rho : r \mapsto (1/h)\sum_{g\in G} g(r)$ sends $R \to R^G$: the sum of the elements in the orbit of R is clearly mapped to itself by every $g_0 \in G$. If $a \in R^G$, $\rho(ar) = a\rho(r)$ since $g(ar) = g(a)g(r) = ag(r)$ for all $g \in G$, and the map clearly preserves sums. Every $a \in R^G$ maps to $(1/h)(ha) = a$, and so ρ is the required splitting. $\square$

(e) If $S = R \oplus W$, we have that $S_m = R_m \oplus W_m$ for every maximal ideal of R. Hence, we may assume that (R, m) is local. Note that if S is a product of R-algebras, say $S = S_1 \times \cdots \times S_k$, where e_i denotes the idempotent with ith coordinate 1 and other coordinates 0, then $1 = e_1 \oplus \cdots \oplus e_k$. Moreover, we may also write $S = S_1 \oplus \cdots \oplus S_k$. Let $\theta : S \to R$ be the R-linear retraction. Then $\sum_{i=1}^k \theta(e_i) = 1$, and so at least one of $\theta(e_i) = u \notin m$, and u is a unit of R. The restriction of θ to $S_i = e_iS$ gives an R-linear map of S_i to R that sends e_i to u, and so multiplying by u^{-1} we obtain an R-linear retraction of S_i to R. Since S is Cohen-Macaulay, so is every S_j. Therefore, we may assume without loss of generality that S is not a product.

Note that S/mS is module-finite over R/mR and therefore 0-dimensional. Thus, the primes of S lying over m are maximal. Also, if Q is maximal in S, $R/(Q \cap R) \to S/Q$ is an injection into a field and is module-finite. Thus, $R/(Q \cap R)$ is a field, and so $Q \cap R = m$. If m consists entirely of zerodivisors on S, it is contain in the union of the associated primes of S, and so is contained in one of them. Since S is Cohen-Macaulay, this associated prime Q must be a minimal prime of S. Thus, Q is a minimal prime of S that lies over m, and so it is also a maximal ideal of S. This means that $\{Q\}$ is both open and closed: every maximal ideal is a closed point, and $\mathrm{Spec}\,(S) - \{Q\}$ is the union of the finitely many closed sets $V(P_i)$ as P_i runs through the minimal primes of S distinct from Q. Thus, S has the form $S_1 \times A$ where A is zero-dimensional. Since S is not a product, this can only happen if $S = A$, and then R is zero-dimensional and, hence, Cohen-Macaulay.

We have now completed the proof in the case where the depth of S on m is 0. Otherwise, we can choose $x \in m$ that is a nonzerodivisor in S. Then S/xS is Cohen-Macaulay, R/xR splits from S/xS, and it follows by induction that R/xR is Cohen-Macaulay. Since R is local and $x \in m$ is not a zerodivisor, it follows that R is Cohen-Macaulay. $\square$

6. It is easy to see that an elementary row or column operation does not affect the ideal generated by the $t \times t$ minors of the matrix. With $x = x_{mn}$ invertible, we may subtract x_{in}/x times the bottom row from the ith row, $1 \le i \le m-1$. Then x_{ij} changes to $y_{ij} = x_{ij} - x_{in}x_{nj}/x$, $1 \le i \le m-1$, $1 \le j \le n-1$, while the bottom row is unchanged and the last column becomes 0 except for the bottom entry x. We may then subtract x_{mj}/x times the last column from the jth column, $1 \le j \le n-1$, and multiply the last row by $1/x$. The matrix has become $\begin{pmatrix} Y & 0 \\ 0 & 1 \end{pmatrix}$ in block form, where $Y = (y_{ij})$ is $(m-1)\times(n-1)$. The $t \times t$ minors that involve 1 correspond bijectively to the $t-1$ size minors of Y, and each, up to sign, equals one of them. Those that do not involve 1 are obviously in $I_{t-1}(Y)$: expand by minors with respect one row or column. Hence, $I_{t-1}(Y)K[X]_x = I_t(X)K[X]_x$. The y_{ij} together with the variables x_{mj}, $1 \le j \le n$, x_{in}, $1 \le i \le n-1$ are algebraically independent over K and together with $1/x = 1/x_{mn}$ generate $K[X]_x$. So $K[X]_x$ is S_x where S is the polynomial ring in the x_{nj} and x_{in} over $K[Y]$. The claim follows. $\square$

7. We show the 2×2 minors are a Gröbner basis using the Buchberger criterion. Thus,

the initial ideal is generated by products of pairs of elements on the "back" diagonals: the *back diagonal* consists of the two elements not on the main diagonal. Since the ideal is graded and x_{mn} is not on a back diagonal, it is not a zerodivisor on the initial ideal, and so is not a zerodivisor on $I_2(X)$. With notation as in 5., one wants to show that $I_1(Y)$ is prime, which is clear: it is generated by a subset of the variables. It remains to apply the Buchberger criterion. If the two minors involve 4 columns or 4 rows, the back diagonals do not overlap, and so the initial terms are relatively prime. If the minors lie in a 2×3 (resp., 3×2) submatrix, the check is the same as in 6. of Problem Set #1: if x is the other variable in the column (resp., row) of the common element of the back diagonals, one gets $\pm x$ times the third 2×2 minor. Let $\Delta_{ab,ij} = x_{ai}x_{bj} - x_{aj}x_{bi}$. We may assume the 2 minors lie in rows indexed $a < b < c$ and columns indexed $i < j < k$, all needed, and that their back diagonals meet in one element. There are five cases not already covered: the pairs of back diagonals (each written as a product) are: (1) $x_{ci}x_{bj}$, $x_{ci}x_{ak}$, (2) $x_{ci}x_{aj}$, $x_{ci}x_{bk}$, (3) $x_{bj}x_{ci}$, $x_{bj}x_{ak}$, (4) $x_{ak}x_{bj}, x_{ak}x_{ci}$, and (5) $x_{ak}x_{bi}, x_{ak}x_{cj}$. The 5 checks, all similar, are:

(1) $\quad x_{ak}\Delta_{bc,ij} - x_{bj}\Delta_{ac,ik} = -x_{bi}\Delta_{ac,jk} - x_{ck}\Delta_{ab,ij}$.

(2) $\quad x_{bk}\Delta_{ac,ij} - x_{aj}\Delta_{bc,ik} = -x_{ai}\Delta_{bc,jk} + x_{ck}\Delta_{ab,ij}$.

(3) $\quad x_{ak}\Delta_{bc,ij} - x_{ci}\Delta_{ab,jk} = x_{aj}\Delta_{bc,ik} - x_{bi}\Delta_{ac,jk}$.

(4) $\quad x_{ci}\Delta_{ab,jk} - x_{bj}\Delta_{ac,ik} = -x_{aj}\Delta_{bc,ik} - x_{ck}\Delta_{ab,ij}$.

(5) $\quad x_{cj}\Delta_{ab,ik} - x_{bi}\Delta_{ac,jk} = -x_{ai}\Delta_{bc,jk} + x_{ck}\Delta_{ab,ij}$.

The initial terms of righthand products don't cancel, so these are standard epxressions. $\square$

8. (a) Suppose one has an infinite strictly decreasing sequence of finite sets. The largest elements are non-increasing and so must be eventually stable. Call the stable value a_1. Recursively, suppose that for $i \le k$ the i th largest elements are eventually stable with stable value a_i. From the point where all are stable on, the $k+1$ st largest elements are non-increasing and so eventually stable as well, say with stable value a_{k+1}. The recursion yields an infinite strictltly decreasing sequence $a_1 > \cdots > a_k > \cdots$, a contradiction. $\square$

(b) At each step in the process, the set of monomials in the current remainder decreases in the sense of part (a), and so the process must terminate. $\square$