Math 615, Fall 2016 **Problem Set #4**
Due: Friday, April 1

1. Let K be a field, and let $X = \bigl(x_{i,j}\bigr)$ be a 2×3 matrix of indeterminates over K. Let R be the polynomial ring in these six variables over K. Order the variables as in Problem 7. of the preceding assignment. Let Δ_j be the 2×2 minor obtained by deleting the j th column of X and taking the determinant. Use hlex and Schreyer's method to find the module N of relations on Δ_1, Δ_2, Δ_3. Prove that N has two minimal generators.

2. Prove that if R is a Cohen-Macaulay $\mathbb{N}$-graded finitely generated K-algebra, where K is a field, then for every integer $k > 0$ the *Veronese subring* $R^{(k)} = \bigoplus_{d=0}^{\infty} [R]_{kd}$ is Cohen-Macaulay.

3. (a) Let R be a finitely generated K-algebra that is Cohen-Macaulay and let L be a field extension of K. Prove that $L \otimes_K R$ is Cohen-Macaulay.

(b) Let K be a field, let R be a Noetherian K-algebra that is Cohen-Macaulay and let S be a finitely generated K-algebra that is Cohen-Macaulay. Show that $R \otimes_K S$ is Cohen-Macaulay.

4. Let R be a finitely generated $\mathbb{N}$-graded K-subalgebra of the polynomial ring $T = K[x_1, \ldots, x_r]$ and let S be a finitely generated $\mathbb{N}$-graded K-subalgebra of the polynomial ring $U = K[y_1, \ldots, y_s]$. Suppose that R is a direct summand of T and S is a direct summand of U. Show that the Segre product R and S over K is also a direct summand of a polynomial ring and, hence, is Cohen-Macaulay. In particular, the Segre product of T and U is Cohen-Macaulay. [This ring is isomorphic with the ring B obtained by taking the quotient of a polynomial ring in rs variables, corresponding to the entries of an $r \times s$ matrix, and dividing out by the ideal generated by the 2×2 minors of the matrix. There is a surjection of B onto the Segre product, and this is a map of integral domains of the same Krull dimension.]

5. Let X and Y be 2×2 matrices of indeterminates over a field K. Let $I_1(XY - YX)$ be the ideal generated by the entries of $XY - YX$. Prove that $K[X, Y]/I_1(XY - YX)$ is a Cohen-Macaulay domain. [Suggestion: this ring can be shown to be a polynomial ring over a domain that we already know to be Cohen-Macaulay.]

N.B. The same question may be asked for every size n. The corresponding ring is known to be a Cohen-Macaulay domain if $n = 3$ (M. Thompson). This is conjectured to be the case in general, but this is an open question.

6. Let R be a Noetherian domain.

(a) Show that if P a prime ideal of R the depth of R_P is one if and only if P is an associated prime of a nonzero principal ideal of R.

(b) Show that R is the intersection of all the local rings $R_P \subseteq \mathrm{frac}\,(R)$ such that P is prime and the depth of R_P is one.

7. (a) Let G be a linear algebraic group over an algebraically closed field K acting on a K-algebra R so that R is a G-module. Let $H \subseteq G$ be a subgroup that is dense in G in the Zariski topology. Show that $R^H = R^G$.

(b) Let H be the group of unitary matrices $\gamma \in \mathrm{GL}(n, \mathbb{C})$ (i.e., $\gamma \in H$ if its inverse is the transpose of its complex conjugate). Find, with proof, the Zariski closure of H in $\mathrm{GL}(n, \mathbb{C})$.

8. A ring R of prime characteristic $p > 0$ is called *F-split* if the Frobenius endomorphism $F : R \to R$ is injective and $F(R) = R^p$ is a direct summand of R as $F(R)$-modules.

(a) A domain R is called *seminormal* if whenever f is in the fraction field of R and $f^2, f^3 \in R$, then $f \in R$. Prove that an F-split domain is seminormal.

(b) Let K be a field of characteristic $p > 0$ which we assume, for simplicity, is perfect. Prove that the polynomial ring $R = K[x_1, \dots, x_n]$ is F-split, and that R/I is F-split if I is generated by square-free monomials.

(c) Let R be a polynomial ring as in part (b), let G be a subgroup of the group of permutations of $x_1, \dots, x_n$, and let G act on R by K-algebra automorphisms that extend its action on the set of variables. Must R^G be F-split? Prove your answer.