Math 615, Winter 2016 **Problem Set #4 Solutions**

1. From 7. of Problem Set #3 with $n = 3$ and $x_{ij} = x_{i+j-1}$ the Δ_i are a Gröbner basis for hlex, and generating relations come from the checks in Buchberger's criterion: the ones coming from cases (1) and (2) of the Problem Set #3 Solutions are $x_{i1}\Delta_1 - x_{i2}\Delta_2 + x_{i3}\Delta_3 = 0$ for $i = 2$, $i = 1$, resp., yield relations $\rho_i = (x_{i1},\, -x_{i2},\, x_{i3})$ for $i = 1, 2$. The third check is not needed here, but we have not proved this. It gives $x_{11}x_{22}\Delta_1 - x_{12}x_{23}\Delta_3 = x_{12}x_{21}\Delta_1 - x_{13}x_{22}\Delta_2$; hence the relation $(\Delta_3,\, 0,\, -\Delta_1) = x_{22}\rho_1 - x_{11}\rho_2$. Thus, ρ_1 and ρ_2 generate. $\square$

2. The Veronese subring contains the k th power of each generator, and so R is integral over over $R^{(k)}$. Since R is finitely generated over $K \subseteq R^{(k)}$, R is module-finite over $R^{(k)}$. This map is split: $\bigoplus_{j \not\equiv 0 \bmod k}[R]_j$ is a complement for $R^{(k0}$ as a module over $R^{(k)}$. The result now follows from Problem **5.**(e) of Problem Set #3. $\square$

3. (a) Since L is free over K, $L \otimes_K R$ is free, and, in particular, faithfully flat over R. Hence, it suffices to show that the fibers over prime ideals P of R are Cohen-Macaulay, and these have the form $\kappa_P \otimes_R L$, where $\kappa_P = R_P/PR_P$ is a field finitely generated over K. Hence, κ_P is a finite algebraic extension of a pure transcendental extension $F = K(t_1, \ldots, t_n)$ of K. Now $T := F \otimes_K L$ is a localization of the polynomial ring $L[t_1, \ldots, t_n]$, and so is a Cohen-Macaulay F-algebra. Since κ_P is a module-finite extension of F, the fiber $\kappa_P \otimes_R L \cong \kappa_P \otimes_F (F \otimes_K L) = \kappa_P \otimes_F T$ is flat and module-finite over T, so that the fibers are now zero-dimensional and, hence, Cohen-Macaulay. $\square$

(b) Since S is free over K, $R \otimes_K S$ is free over R. It therefore suffices to show that the fibers $\kappa_P \otimes_R (R \otimes_K S) \cong \kappa_P \otimes_K S$ are Cohen-Macaulay. This holds by part (a) (let S play the role of R, and let $L = \kappa_P$). $\square$

4. Let $T = R \oplus M$, where M is an R-module, and $U = S \oplus N$, where N is an S-module. Then $T \otimes_K U \cong (R \otimes_K S) \oplus \big((R \otimes_K N) \oplus (M \otimes_K S) \oplus (M \otimes_K N)\big)$, where each of the four terms is an $(R \otimes_K S)$-module. Hence, $R \otimes_K S$ is a direct summand of $T \otimes_K U$, which is a polynomial ring over K. $\square$

5. Let $X' = X - x_{11}\boldsymbol{I}$ and $Y' = Y - y_{11}\boldsymbol{I}$. Then $XY - YX = X'Y' - Y'X'$, and $K[X, Y] = K[X', Y'][x_{11}, y_{11}]$. If we use $x'_{22} = x_{22} - x_{11}$ and $y'_{22} = y_{22} - y_{11}$, we have that $K[X, Y]/I_1(XY - YX) \cong \big(K[X', Y']/I_1(X'Y' - Y'X')\big)[x_{11}, y_{11}]$. Here, X', Y' are matrices of indeterminates except that there are 0s in the upper left corners. But $I_1(X'Y' - Y'X')$ is $I_2(Z)$ where Z is the 2×3 matrix with rows x_{12}, x_{21}, x'_{22} and y_{12}, y_{21}, y'_{22}.

6. (a) Let f be any nonzero element of P. Then f is a nonzerodivisor in R_P, and so R_P has depth one if and only if R_P/fR_P has depth 0, i.e., if and only if PR_P is an associated prime of the ideal fR_P (equivalently, of the module R_P/fR_P). This is true if and only if P is an associated prime of the ideal fR (equivalently, of the module R/fR), since, quite generally, P is an associated prime of M if and only if PR_P is an associated prime of M_P over R_P. (If $R/P \to M$ is injective, this is preserved when one localizes at P.) Conversely, if u/w has annihilator PR_P, where $u \in M$ and $w \in R - P$, then, since $1/w$ is a unit, $P(u/1) = 0$ in M_P. This implies that for some $v \in R - P$, $Pvu = 0$ in M, and then P is the annihilator of vu.) $\square$

(b) The intersection clearly contains R. Suppose $g, f \in R$, $f \neq 0$, and $g/f \notin R$. Since $g \notin fR$, if $\mathfrak{A}_1 \cap \cdots \cap \mathfrak{A}_h$ is a primary decomoposition of fR, we can choose $\mathfrak{A}_i$ such that $g \notin \mathfrak{A}_i$. Let P be the radical of $\mathfrak{A}_i$, which is an associated prime of f. Then R_P has depth 1, and $g \notin \mathfrak{A}_i R_P$, so that $g \notin fR_P$ and $g/f \notin R_P$. $\square$

7. (a) Let $f \in R^H$ and suppose $f \in V \subseteq R$ where V is finite-dimensional as a K-vector space and G-stable. We have a regular map $G \to V \times V$ via $\gamma \mapsto \big(\gamma(v), v\big)$. The inverse image of the diagonal $\Delta \subseteq V \times V$ is closed, and contains H. Since H is dense, it contains G. The fact that G maps into Δ implies that all of G fixes f. $\square$

(b) Let $\overline{H}$ be the closure of the unitary matrices H. We will show that $\overline{H}$ is the general linear group G. Fix $h \in H$. Left multiplication by h is an automorphism of G as an algebraic set that stabilizes H. Hence, it stabilizes $\overline{H}$, and $H\overline{H} \subseteq \overline{H}$. Fix $g \in \overline{H}$. Then right multiplication by g^{-1} is an automorphism of G as an algebraic set and since $\overline{H}g^{-1}$ is a closed set containing H, it contains $\overline{H}$. Hence, $\overline{H}\,\overline{H} \subseteq \overline{H}$. Likewise, $g \mapsto g^{-1}$ is an automorphism of the algebraic set G that stabilizes H, and so it stabilizes $\overline{H}$. Hence, $\overline{H}$ is a subgroup of G.

To show that $\overline{H} = G$, it suffices to show $\overline{H}$ contains the diagonal and elementary matrices. Let D_1 be the set of diagonal matrices with an element of $\mathbb{C} - \{0\}$ in the 1, 1 spot and 1 elsewhere on the diagonal. Since the elements of absolute value 1 are Zariski dense in $\mathbb{C} - \{0\}$ (any infinite set is dense), $\overline{H}$ contains D_1, and similarly, all diagonal matrices such that diagonal entries are 1 except in one spot. Since $\overline{H}$ is a group, $\overline{H}$ contains all diagonal matrices. Since an elementary matrix is the direct sum of an identity matrix with a 2×2 matrix, whether the elementary matrices are in $\overline{H}$ reduces to the case $n = 2$. Orthogonal matrices are unitary, and symmetric matrices are orthogonally conjugate to diagonal matrices. Thus, $\left(\begin{smallmatrix}1 & 1\\ 1 & 0\end{smallmatrix}\right) \in \overline{H}$ and $A = \left(\begin{smallmatrix}1 & 1\\ 1 & 0\end{smallmatrix}\right)\left(\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right) = \left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right) \in \overline{H}$, so that $\left(\begin{smallmatrix}1 & 0\\ 0 & c^{-1}\end{smallmatrix}\right)A\left(\begin{smallmatrix}1 & 0\\ 0 & c\end{smallmatrix}\right) \in \overline{H}$, $c \neq 0$, and this is a typical upper triangular elementary matrix. Lower triangular elementary matrices can be obtained similarly. (Alternatively one may use the fact that every invertible matrix is the product of a unitary matrix and a Hermitian matrix.) $\square$

8. (a) If f^2, $f^3 \in R$ then $f^n \in R$ for all $n \geq 2$ in $\mathbb{N}$. Hence, $f^p \in R$, whence under $F : R \to R = S$, $F(f^3) \in F(f^2)S$. Since F is split, ideals are contracted, and we have that $f^3 \in f^2R$ so that $f \in R$. $\square$

(b) If R is a polynomial ring over a perfect field K, a polynomial is a pth power iff all exponents are divisible by p. $F(R) \subseteq R$ has a complement the K-span of all monomials such that at least one exponent occurring is not divisible by p. Call the corresponding splitting ϕ. The same description, applied to monomials that are nonzero mod I, gives a splitting of $F(R/I) \to R/I$ when I is generated by square-free monomials. $\square$

(c) G acts on monomials, and a K-basis for R^G consists of sums of orbits. One monomial in an orbit has all exponents divisible by p if and only if all the monomials occurring do. It follows that ϕ maps R^G into $F(R^G)$ and this implies that it splits $F(R^G) \subseteq R^G$. $\square$