Math 615, Fall 2016 **Problem Set #5**
Due: Friday, April 15

1. Let $R$ be a Noetherian domain of prime characteristic $p > 0$, $I$, $J \subseteq R$, and let $f \in R$.

(a) Show that if $I$ is tightly closed, then $I :_R J$ is tightly closed.

(b) Show that if $f \neq 0$ and $g \in (f)^*$, then $g/f$ is integral over $R$. (You may assume the fact that the integral closure of $R$ is an intersection of Noetherian discrete valuation rings.)

(c) Show that if $R$ is integrally closed, then $(fI)^* = f(I^*)$.

2. Let $(R, m, K)$ be a Cohen-Macaulay ring of dimension $n$. Let $\underline{x} = x_1, \dots, x_n$ and $\underline{y} = y_1, \dots, y_n$ be two systems of parameters. Let $\mathfrak{A}_{\underline{x}}$ be the annihilator of $m$ in $R/(\underline{x})R$, with similar notation for $\underline{y}$ and other systems of parameters.

(a) If $n = 1$, prove that multiplication by $y_1$ on the numerators induces an injective map $R/x_1R \to R/x_1y_1R$ that carries $\mathfrak{A}_{x_1} \cong \mathfrak{A}_{x_1y_1}$. Hence, by symmetry, $\mathfrak{A}_{x_1} \cong \mathfrak{A}_{y_1}$.

(b) Show that $\mathfrak{A}_{\underline{x}} \cong \mathfrak{A}_{\underline{y}}$ in general. (You may use that any two systems of parameters are joined by a finite chain in which consecutive systems differ in only one element.)

(c) If $y_i = x_i^{t_i}$ with $t_i \geq 1$ for $1 \leq i \leq n$, show that the map $R/(\underline{x})R \to R/(\underline{y})R$ induced by multiplication by $x_1^{t_1-1} \cdots x_n^{t_n-1}$ on numerators induces an injection that carries $\mathfrak{A}_{\underline{x}} \cong \mathfrak{A}_{\underline{y}}$.

[$\dim_K(\mathfrak{A}_{\underline{x}})$ is independent of the choice of $\underline{x}$ and is called the *type* of the Cohen-Macaulay local ring $R$. A Cohen-Macaulay local ring of type one is called *Gorenstein*.]

3. Let $(R, m, K)$ be a Cohen-Macaulay local domain of prime characteristic $p > 0$. Let $x_1, \dots, x_n$ be a system of parameters. Suppose that $(x_1, \dots, x_n)R$ is tightly closed. Prove that for every system of parameters $y_1, \dots, y_n$, the ideal $(y_1, \dots, y_n)R$ is tightly closed.

4. Let $\underline{x}^- = x_1, \dots, x_{n-1} \in R$. Let $M$ be any $R$-module. Show that there is a long exact sequence of Koszul homology a typical part of which is

$$\cdots \to H_{i+1}(\underline{x}^-, z; M) \to H_i(\underline{x}^-, y;\, M) \to H_i(\underline{x}^-, yz;\, M) \to H_i(\underline{x}^-, z;\, M) \to \cdots .$$

[Suggestion: reduce to the situation where $R$ is replaced by $B = A[X_1, \dots, X_{n-1}, Y, Z]$ and make use of a short exact sequence of $B$-modules.]

5. Let $R$ be be the local ring $K[[x, y]]/(x^2, xy)$ where $x$ and $y$ are formal power series indeterminates, and $K$ is a field. Let $m$ denote the maximal ideal of $R$. Determine the minimal modules of syzygies of $K = R/m$ as a module over $R$: show that each is a direct sum of copies of $K$ and $m$. Determine, for all $i$, the dimension of $\mathrm{Tor}_i^R(K, K)$ as a vector space over $K$.

6. Let $R$ be the ring $K[x^2, xy, y^2] \subseteq K[x, y] = S$. Then $S = R \oplus M$, where $M$ is the $R$-submodule $xR + yR$ of $S$ generated by $x$ and $y$. Note that $M$ is isomorphic as a module to both $(x^2, xy)R$ and to $(xy, y^2)R$. Show that all of the minimal modules of syzygies of $M$ are isomorphic to $M$, and determine the modules $\mathrm{Tor}_i^R(M, M)$ for all $i \geq 1$.

7. (a) Let $(A, m)$ be a local ring and let $\underline{x} = x_1, \dots, x_n \in m$ generate a primary ideal. For any finitely generated $A$-module $M$, define $\chi(\underline{x}; M) = \sum_{i=0}^{n} (-1)^i \ell\big(H_i(\underline{x};\, M)\big)$. Show that if $0 \to M' \to M \to M'' \to 0$ is exact, then $\chi(\underline{x}, M) = \chi(\underline{x}; M') + \chi(\underline{x}, M'')$.

(b) Prove that if $n \geq 2$ and $M$ has finite length, then $\chi(\underline{x}; M) = 0$.

(c) Prove that if $(R, m)$ is an Artin ring and $x_1, x_2 \in M$, then $H_1(x_1, x_2; R)$ is never a cyclic module over $R$, i.e., requires at least two generators.

8. Fix $n \geq 2$ and integers $a_1, \ldots, a_n > 0$. Let $K$ be a field of characteristic $p > 0$ ($p$ will vary) and let $f = x_1^{a_1} + \cdots + x_n^{a_n}$. Let $R = K[x_1, \ldots, x_n]/(f)$, and let $I = (x_1, \ldots, x_{n-1})R$. Let $\alpha = \frac{1}{a_1} + \cdots + \frac{1}{a_n}$. Show that if $\alpha \leq 1$, then $x_n^{a_n - 1}$ is in the tight closure $I^*$ of $I$. Show that if $\alpha > 1$ then for all sufficiently large primes $p$, the element $x_n^{a_n - 1} \notin I^*$.