Math 615, Winter 2016 **Problem Set #5 Solutions**

1. (a) If $u \in (I :_R J)^*$, there exists $c \neq 0$ in R such that for all $q \gg 0$, $cu^q \in (I :_R J)^{[q]} \subseteq I^{[q]} :_R J^{[q]}$. Hence, if $j \in J$, for all $q \gg 0$, $j^q(cu^q) = c(ju)^q \in I^{[q]}$, so that $ju \in I^* = I$. Since this holds for all $j \in J$, $u \in I :_R J$. $\square$

(b) If this is false, we can find a Noetherian discrete valuation ring $V \supseteq R$ such that $f/g \notin V$. Since, in V, $f \in (gV)^*$, it will suffice to show that gV is tightly closed. If $g = 0$ this is clear. Otherwise, choose $cd \neq 0$ such that $cf^q \in g^qV$ for all $q \gg 0$. Then $\mathrm{ord}\,(c) + q\,\mathrm{ord}\,(f) \geq q\,\mathrm{ord}\,(g)$ and $\mathrm{ord}\,(f) \geq \mathrm{ord}\,(f) - \mathrm{ord}\,(c)/q$ for all $q \gg 0$. It follows that $\mathrm{ord}\,(f) \geq \mathrm{ord}\,(g)$, as required. $\square$

(c) If $f = 0$, this is clear. Assume $f \neq 0$. $(fI)^* \subseteq (fR)^* = fR$ by (b). But if $r \in R$, $fr \in (fI)^*$ iff for some nonzero c and all $q \gg 0$, $c(fr)^q \in (fI)^{[q]} = f^qI^{[q]}$, i.e., $cr^q \in I^{[q]}$. Thus, $fr \in (fI)^*$ iff $r \in I^*$, and the result follows. $\square$

2. (a) We omit the subscript $\;_1$. Multiplcation by y carries R isomorphically onto yR (since y is a nonzerodivisor) and $xR \subseteq R$ isomorphically onto xyR. Hence, the map induced by multiplication by y carries R/xR isomorphically on $yR/xyR \subseteq R/xyR$. In particular, the specified map is an injection, and must take $\mathfrak{A}_x = \mathrm{Ann}_{R/xR}m$ into $\mathfrak{A}_{xy} = \mathrm{Ann}_{R/xyR}m$. Suppose u in R is such that $mu \subseteq xyR$, and $\overline{u}$ is its image in R/xyR. Then $xu \in xyR$, and since x is a nonzerodivisor, $u \in yR$. Hence, $\overline{u} \in \mathrm{Ann}yR/xyRm$, and the isomorphism $R/xR \cong yR/xyR$ shows that $\overline{u}$ must be the image of an element of $\mathfrak{A}_x$. $\square$

(b) Using the parenthetical comment, we may reduce the problem at once to the case where the two systems are the same except for one element. Since the order of the parameters is not relevant, we may assume that the systems are $x_1, \ldots, x_{n-1}, x$ and $x_1, \ldots, x_{n-1}y$. The issue is unaffected if we replace R by $\overline{R} = R/(x_1, \ldots, x_{n-1})R$ and x, y their images in $\overline{R}$, each of which is a parameter for the one-dimensional Cohen-Macaulay ring $\overline{R}$. The result is then immediate from part (a). $\square$

(c) It suffices to prove the result when all the t_j except t_i are equal to one: one may then apply that result n times, one time for each x_j, to obtain the desired fact. We may proceed by placing x_j last in the sequence, and working modulo the other elements of the sequence, as in part (b), to reduce to the case where $n = 1$. The result then follows from taking x to be the image of x_j and y to be $x_j^{t_j-1}$. $\square$

3. As in Problem 2., it suffices to consider the case where the two systems of parameters are the same except for one element, and one can compare each of $I = x_1, \ldots, x_{n-1}, x$ and $x_1, \ldots, x_{n-1}, y$ with $J = x_1, \ldots, x_{n-1}, xy$. If an m-primary ideal I is not tightly closed, let u be an element of the tight closure not in I. Then either $mu \subseteq I$, are we can choose $z_1 \in m$ such that $z_1u \notin I$, while it is still true that $z_1u \in I^*$. Repeating this process finitely many times, we can find an element $u \in I^* - I$ such that $mu \subseteq I$: the process cannot continue more than N steps, where $m^N \subseteq I$. Thus, I fails to be tightly closed if and only if there is an element of $I :_R m$ that is in $I^* - I$. By the result of Problem 2., we have a bijection between $(I : m)/I$ and $(J : m)/I$ induced by multiplication by y. Thus, it suffices to show for $u \in I : m$ that $u \in I^*$ if and only if $yu \in J^*$. But $c(yu)^q \in (x_1^q, \ldots, x_{n-1}^q, x^qy^q)$ iff $cu^q \in (x_1^q, \ldots, x_{n-1}^q, x^qy^q) : y^q = (x_1^q, \ldots, x_{n-1}^q, x^q)$ (one may check this working mod

$\mathfrak{A} = (x_1^q, \ldots, x_{n-1}^q)$, where it follows from the fact that x and y are both nonzero divisors mod $\mathfrak{A}$), and the result is now immediate.

4. As in the bracketed suggestion, there is a short exact sequence

$$0 \to B/(X_1, \ldots, X_{n-1}, Z) \xrightarrow{f} B/(X_1, \ldots, X_{n-1}, YZ) \xrightarrow{g} B/(X_1, \ldots, X_{n-1}, Y) \to 0$$

where g is the quotient surjection and f is induced by multiplication by Y on the numerators. Note that each ideal in a denominator is generated by a regular sequence in B. If $C = B/(X_1, \ldots, X_{n-1}) \cong \mathbb{Z}[X, Y]$, this is simply the short exact sequence $0 \to C/ZC \to C/YZC \to C/YC \to 0$ that one gets by noting that $YC/YZC \cong C/ZC$.

Now apply $\_ \otimes_B M$. The long exact sequence for Tor gives the required result, once one notes that for each quotient, the Koszul complex gives a free resolution, so that every Tor in the long exact sequence can be viewed instead as a Koszul homology module.

5. If we take minimal generators for A and minimal generators for B, together they give minimal generators for $A \oplus B$, and we find that a minimal module of syzygies for $A \oplus B$ can be obtained as the direct sum of minimal modules of syzygies for A and B.

The first minimal module of syzgies of K is m. To obtain the second minimal module of syzygies, note that $ax + by = 0$ implies that a and b are both in m (or else x and y are not minimal generators of m. But m kills x, so that this is equivalent to $a \in m$ and $by = 0$, and then $b \in xR \cong K$. Thus, the second module of syzygies is $m \oplus K$. Suppose that we have shown the i th module of syzygies is isomorphic to $K^{a_i} \oplus m^{b_i}$. Then the $i+1$ st module of syzygies is $m_i^a \oplus (K \oplus m)^{b_i} \cong K^{b_i} \oplus m^{a_i + b_i}$. It follows by a straightforward induction on i that the minimal i th module of syzgies of K is $K^{f_{i-1}} \oplus m^{f_i}$, where f_i is the i th Fibonacci number, determined recursively by the rules $f_0 = f_1 = 1$ and $f_{i+1} = f_i + f_{i-1}$ for $i \geq 1$. The rank of the i th free module in a minimal resolution is the same as the least number of generators of the i th module of syzgies, which $f_{i-1} + 2f_i = f_{i-1} + f_i + f_i = f_{i+1} + f_i = f_{i+2}$, and this is the same as the dimension of $\mathrm{Tor}_i^R(K, K)$ as K-vector space.

6. If we map $R \oplus R \twoheadrightarrow M$ so that $(r, s) \mapsto rx + sy$ the kernel consists of $\{(r, s) \in R^2 : rx + sy = 0\}$. This implies that in $K[x, y]$, (r, s) is a multiple of $v = (y, -x)$, which implies that the kernel is $(Rx + Ry)v \cong M$ It follows that all the minimal modules of syzygies are $\cong M$, and that the resolution has the form $\cdots \to R^2 \xrightarrow{A} R^2 \xrightarrow{A} R^2 \to \cdots \to R^2 \xrightarrow{A}> R^2 \to M \to 0$, where the matrix of A is $w(x\ y)$, i.e., has columns xw and yw, where w is the transpose of v, i.e., v written as a column. Thus, for $i \geq 1$, $\mathrm{Tor}_i^R(M, M) \cong \mathrm{Ker}\,(h)/\mathrm{Im}\,(h)$, where h is the map $M^2 \to M^2$ induced by A. $\begin{pmatrix} a \\ b \end{pmatrix} \in M^2$ is in the kernel iff $xa + yb = 0$, i.e., it is an element of M^2 that has the form cw, where $c \in R$. The image of M^2 is all elements of the form $w(xm_1 + ym_2)$ where $m_1, m_2 \in xR + yR$, and this yields $w(x^2, xy, y^2)R$. The quotient is a copy of K spanned by the image of w, so that $\mathrm{Tor}_i^R(M, M) \cong K$, $i \geq 1$.

7. (a) This follows from taking the alternating sum of all the lengths in the long exact sequence for Koszul homology (there are only finitely many nonzero terms). $\square$

(b) By part (a), this reduces to the case where $M = K$. If any of the x_i is a unit, then all the Koszul homology vanishes, since the x_i kill the Koszul homology. Therefore, we may assume that all the x_i are in the maximal ideal, so that their action on K is the same as multiplication by 0, the 0 map. Hence, if $n \geq 1$ (not just 2), the Koszul homology for K is identical with the Koszul complex, and the alternating sum of the lengths is $\sum_{j=0}^{n}(-1)^j\binom{n}{j}$, which is the same as $\bigl(1+(-1)\bigr)^n = 0$. $\square$

(c) Let h_i be the length of $H_i(x_1, x_2; A)$. By part (a), $h_1 = h_0 + h_2$. Since h_2 is the length of annihilator of (x_1, x_2) in A, it is positive. Thus, $h_1 > h_0$. If h_1 is cyclic, it would be a homomorphic image of $A/(x_1, x_2)$, which has length h_0, since it is a cyclic module killed by x_1 and x_2. Thus, $h_0 \leq h_1$, a contradiction. $\square$

8. The issue is whether we can choose $c \in R - \{0\}$ such that $c(x_n^{a_n-1})^q \in I^{[q]} = (x_1^{a_1 q}, \dots, x_{n-1}^{a_{n-1} q})R$ for all $q = p^e \gg 0$. Let $a = a_n$. If we write $q = sa - r$ with $1 \leq r < a$, so that $s/q = 1/a + r/aq$ with $r/a < 1$. we can rewrite this as $c(x_n^{a_n})^{q-s} x_n^r \in I^{[q]}$, and since $1, x_n, \dots, x_n^{a-1}$ is a free basis for R over the polynomial subring $T = K[x_1, \dots, x_{n-1}]$ and $x_n^{a_n}$ is the same, up to sign, as $g = x_1^{a_1} + \cdots + x_{n-1}^{a_{n-1}} \in T$, this is equivalent to whether we can choose $c \in R - \{0\}$ such $cg^{q-s} \in I_q := (x_1^q, \dots, x_{n-1}^q)T$ for all $q > 0$. Note that $I^{[q]} = I_q R$. Suppose that we take $c = 1$ if $\alpha < 1$ and if $\alpha = 1$ we take $c = x_1^{d_1} \cdots x_{n-1}^{d_n}$ where the d_i are chosen so that $r/a < \sum_{i=1}^{n-1} d_i/a_i$. First suppose $\alpha \leq 1$. For every partition $q - s = b_1 + \cdots + b_{n-1}$, if any $d_i + a_i b_i \geq q$ we have that the corresponding term in the multinomial expansion of cg^{n-s} is in $I^{[q]}$. Otherwise, $b_i < q/a_i$, and then $q - s = \sum_{i=1}^{n-1} b_i < \sum_{i=1}^{n-1}(q/a_i - d_i/a_i)$, and then $1 \leq s/q - \sum_{i=1}^{n-1} d_i/a_i)/q + \sum_{i=1}^{n-1} 1/a_i$. $1 \leq (r/a - \sum d_i/a_i)(1/q) + \alpha$. If $\alpha < 1$, this will be false for $q \gg 0$. $c = 1$ this will be false for all q, since the first term on the right is negative. It remains to show that we cannot choose c for $\alpha > 1$ and $p \gg 0$. in fact, we shall show this whenever $p > (n+2)/(\alpha - 1)$ and $4a_n$, which we assume from now on.

We may replace K by its algebraic closure: if there is no value of c that works in the larger domain, there is clearly no value that works in the original ring. We may assume that $p \gg 0$ does not divide any a_i, and, in particular, we may assume that $p > a_n$. We next show that if $c \in R - \{0\}$ and $cu^q \in J^{[q]}$, then we can take replace c by x_n^{2a-2} and the same statement holds. Let $T = K[x_1, \dots, x_{n-1}]$. Since R is integral over T, the element c has a multiple in $T - 0$ (the constant coefficient of a least degree monic polynomial over T that c satisfies), and so we may assume without lost of generality that $c \in T$. The monomials in $x_1^{1/p}, \dots, x_n^{1/p}$ with all exponents at most $p-1$ form a free basis for $T^{1/p}$ over T. Since x_n is a separable element over the fraction field of T, the same monomials span a free R-module $G_1 \subseteq R^{1/p}$ (in fact, G_1 is a ring, and is the same as $R[T^{1/p}]$, although we won't use this). We next note that x_n^{a-1} multiplies $R^{1/p}$ into G_1. It suffices to show that $x_n^{a-1} x_n^{i/p} \in G_1$ for $0 \leq i \leq p-1$, i.e., that $((x_n)^{1/p})^{ap-p+i)} \in G_1$, $0 \leq i \leq p-1$. Since $(x_n^{1/p})^a \in G_1$ (it is the p th root of x_n^a, which is equal to an element of T), and $(x_n^{1/p})^p = x_n \in G_1$, it is enough to show that each $ap - p + i$ is in the semigroup generated by a and p. Since p is invertible mod a, one the elements $0, p, 2p \dots, (a-1)p$ has the same residue as i mod $a-1$, say jp, and then $ap - p + i - jp$ is a nonnegative multiple of a, say k, which yields the result since $ap - p + i = jp + ak$.

Let G_e denote the R-span of the set $\mathcal{M}_e$ of monomials in $x_1^{1/p^e}, \dots, x_{n-1}^{1/p^e}$ such that every exponent occurring in the monomial is at most $p^e - 1$. (Again, $G_e = R[B^{1/p^e}]$.) Exactly as in the case $n = 1$, G_n is R-free on these generators. Let

$$\gamma_e = (a-1)(1 + 1/p + 1/p^2 + \cdots + 1/p^{e-1}).$$

We next show by induction on n that $x_n^{\gamma_e} R^{1/p^e} \subseteq G_e$. We have already proved the case where $e = 1$. Assuming the result for a certain e, we may take p th roots to obtain that $x_n^{\gamma_e/p} R^{1/p^e}$ is in the $R^{1/p}$-span of the monomials $\{\mu^{1/p} : \mu \in \mathcal{M}_e\}$. If we multiply by x_n^{a-1}, we obtain that $x_n^{a-1+(\gamma_e/p)} R^{1/p^{e+1}}$ is the R-span of the monomials $\{\nu\mu^{1/p} : \nu \in \mathcal{M}_1,\ \mu \in \mathcal{M}_e\} = \mathcal{M}_{e+1}$, since we may replacing each $x_n R^{1/p}$ by the R-span of the $\nu \in \mathcal{M}_1$ using the result for $e = 1$. Since $a - 1 + (\gamma_e/p) = \gamma_{e+1}$, the result follows. Since $\gamma_e < 2$ for all e, we can conclude that for all $q = p^e$, $x_n^{2(a-1)} R^{1/q} \subseteq G_e$, a free R-module. We write $d = x_n^{2(a-1)}$ for brevity.

Now suppose that $cu^q \in J^{[q]}$ for all $q \gg 0$, where $c \in B - \{0\}$. Let $q' = p^{e'}$. Then $cu^{qq'} \in (J^{[q]})^{[q']}$ for all $q' \gg 0$, and $c^{1/q'} u^q \in J^{[q]} R^{1/q'}$. This yields $c^{1/q'} du^q \in J^{[q]} G_{e'}$ for all $q' \gg 0$, since $dR^{1/q'} \subseteq G_{e'}$, and so $c^{1/q'} \in J^{[q]} G'_e :_{G_{e'}} du^q$ for all q'. Note that $c^{1/q'} \in G_{e'}$ because $c \in T$. Because $G_{e'}$ is R-free, this is the same as $(J^{[q]} :_R du^q) G_e$ which is contained in $(J^{[q]} :_R du^q) R^{1/q'}$. Taking q' th powers, we find that for all $q' \gg 0$, $c \in (J^{[q]} :_R du)^{[q']}$. Since $c \neq 0$, this is a contradiction unless$J^{[q]} :_R du^q = R$ (in a Noetherian domain, the intersection of the powers of a proper ideal is 0, since any element will also be in all the powers of the maximal ideal of the local domain obtained by localizing at a maximal ideal that contains it). But this means that $du^q \in J^{[q]}$.

Hence, if there is an choice of $c \neq 0$ such that $c(x_n^{a-1})^q \in I^{[q]}$ for all $q \gg 0$, we may take c to $x_n^{2(a-1)}$. That is we need only show that we cannot have $cx_n^{(a-1)(q+2)} \in I^{[q]}$ for $q \gg 0$. We have $(a-1)(q+2) = a(q+2) - (q+2)$. Write $q + 2 = va - r$, where $0 \leq r \leq a - 1$. Then the issue is whether $(x_n^a)^{q-v+2} x_n^r \in I^{[q]}$, and, as earlier, this is equivalent to asking whether $g^{q-v+2} \in (x_1^{qa_1}, \ldots, x_{n-1}^{qa_n})B =: I_q$. To prove this does not hold, it suffices to show that there is a positive integer $s \geq q - v + 2$ and integers $s_1, \ldots, s_{n-1} \in \mathbb{N}$ such that $s_i a_i < q$ for $1 \leq i \leq n-1$, $s_1 + \cdots + s_{n-1} = s$, and the multinomial coefficient $\binom{s}{s_1, \ldots, s_{n-1}}$ is not divisible by p. If these conditions hold, then g^s is not in $I^{[q]}$, for the multinomial expansion of g^s will have an $\binom{s}{s_1, \ldots, s_{n-1}} \prod_{i=1}^{n-1} (x_i^{a_i})^{s_i}$ term that is not in I_q. Since $s \geq q - v + 2$, we also have that $g^{q-v+2} \notin I_q$, as required.

We shall choose all of the s_i to have the form $t_i p^{e-1}$ where $0 \leq t_i \leq p - 1$ and, moreover, $t = t_1 + \cdots + t_{n-1} \leq p - 1$. Let $t = \lceil (q - v + 2)/p^{e-1}) \rceil$, and let $w_i = \lfloor (p/a_i) \rfloor < p/a_i$. First observe that t is at most one more than $(q - v + 2)/p^{e-1} = p + 2/p^{e+1} - v/p^{e-1}$, where $v = (q + 1 + r)/a$, so that $v/p^{e-1} = (p/a) + (1 + r)/p^{e-1}a$. Thus, $t \leq 1 + p - p/a + 2/p^{e-1} + (1 + r)/p^{e-1}a \leq (1 - 1/a)p + 1 + 3/p^{e-1}$, which is smaller than p provided $p/a > 4$.

Note that $a_i(w_i p^{e-1}) < a_i(p/a_i)p^{e-1} = q$, and that $wp^{e-1} \geq q - v + 2$. We do not have that $t = w_1 + \cdots + w_{n-1}$, but we have, in fact, that $w_1 + \cdots + w_{n-1} \geq t$. To see this, note that $t \leq \big(q - v + 2)/p^{e-1}\big) + 1$ and that $w_i \geq (p/a_i) - 1$. Hence, it will suffice if $(\sum_{i=1}^n p/a_i) - (n-1) \geq p + 1 - (v-2)/p^{e-1}$. But $v = (q + 2 + r)/a$, so, after we divide by p, this comes down to the inequality $-(n-1)/p + \sum_{i=1}^{n-1} 1/a_i \geq 1 + (1/p) - (q + 2 + r)/aq + 2/q$. This becomes $\sum_{i=1}^{n-1} 1/a_i \geq (n-1)/p + 1 + (1/p) - (2 + r)/aq + 2/q$. The right hand side is at most $n/p + 2/q \leq (n+2)p$ Hence, if $p \geq (n+2)/(\alpha - 1)$, we have that the $w_1 + \cdots + w_{n-1} \leq w$.

Therefore, by decreasing the w_i suitably, we can choose nonnegative integers t_i such that $t_i \leq w_i$ and $\sum_{i=1}^{n-1} t_i = t$.
We now take $s_i = t_i p^{e-1}$ and $s = tp^{e-1}$. The argument will be complete if we can show that the multinomial coefficient $\binom{s}{s_1, \dots, s_{n-1}}$ does not vanish. This is true because it is equal to the multinomial coefficient $\binom{t}{t_1, \dots, t_{n-1}}$. One can see this by expanding $(z_1 + \cdots + z_{n-1})^{tp^e}$ in two ways: on the one hand, the coefficient of $z_1^{t_1 p^{e-1}} \cdots z_{n-1}^{t_{n-1} p^{e-1}}$ is $\binom{s}{s_1, \dots, s_{n-1}}$. On the other hand, we can think of the expansion as $((z_1 + \cdots + z_{n-1})^t)^{p^{e-1}}$, which gives the coefficient of the required term as $\binom{t}{t_1, \dots, t_{n-1}}^{p^{e-1}}$, which is the same as $\binom{t}{t_1, \dots, t_{n-1}}$. But now we can see that this does not vanish, simply because $t < p$. $\square$