Math 615, Winter 2017 **Problem Set #1: Solutions**

1. It is clear that $R/m \cong K$, so that m is maximal. In R_m, every $1 - x_j$ is a unit, and if $i < j$, the relation $x_i(1 - x_j) = 0$ then implies that $x_i/1 = 0$ in R_m. Thus, $R_m \cong R_m/mR_m \cong K$, and it follows that m is minimal. If $\{m\}$ is open, there is some element $f \notin m$ such that $D(f) = \{m\}$. But f can involve only finitely many x_j: say $f = c + c_1x_1 + \cdots + c_nx_n$ where $c \in K - \{0\}$ and $c_i \in K$, $1 \le i \le n$. There is a homomorphism $R \to K$ such that $x_i \mapsto 0$, $1 \le i \le n$ and $x_i \mapsto 1$, $i > n$: this map preserves the relations $x_i = x_ix_j$ for $i \le j$. The kernel is a maximal ideal in $D(f)$ and is different from m, a contradiction.

EXTRA CREDIT 1. Consider any prime ideal P of R. If $x_h \notin P$, then $x_h(1 - x_i) = 0$ for $h \le i$, which shows that $1 - x_i \in P$ for all $i \ge h$. Thus, the elements x_j such that $x_j \notin P$ form an upper interval, and for every such j, $x_j - 1 \in P$. Except for $P = m$, at least one x_i is not in P, and there will be a least such, say x_n. Then $x_h \in P$ for all $i < n$, while $x_i - 1 \in P$ for all $i \ge n$. Call this prime m_n. The quotient R/m_n is just K, since in the quotient every x_i is identified either with 0 or 1. Thus, every prime of R is maximal, and is one of the maximal ideals in the set $\{m\} \cup \{m_1, m_2, m_3, \ldots, m_n, \ldots\}$. If $f = (1 - x_{n-1})x_n = x_n - x_{n-1}$, $n \ge 2$, then $D(f) = \{m_n\}$ is open (as well as closed), as is $m_1 = D(x_1)$. Also, every open set that contains m has finite complement: it contains $D(g)$ for some $g = c+c_1x_1+\cdots+c_nx_n$, where $c \in K-\{0\}$ and all other $c_i \in K$,and we have that $D(g) \supseteq \{m_i : i > n\}$. Hence, $\mathrm{Spec}\,(R)$ is the one point compactification of a countably infinite discrete space. Thus, X is homeomorphic with $\{0, 1, 1/2, 1/3, \ldots, 1/n, \ldots\}$ if we let m_n correspond to $1/n$ for all $n \ge 1$ and let m correspond to 0.

2. Since $u^m \in R$, S is integral over R, and, by the lying over theorem, the map is surjective. The scheme-theoretic fiber $T = \kappa \otimes_R S$, where $\kappa = R_P/PR_P$, is then nonzero, and $T = \kappa[v]$, where v is the image of u. By the Euclidean algorithm, there are integers a, b such that $am + bn = 1$. We must show $\mathrm{Spec}\,(T)$ has at most one point. If $v^m = 0$ or $v^n = 0$ in K, killing v does not effect $\mathrm{Spec}\,(T)$, which will be the same as $\mathrm{Spec}\,(\kappa)$. If both are not 0, v is a unit and $v = (v^m)^a(v^n)^b \in \kappa$, so that $T = \kappa[v] = \kappa$. $\square$

3. (a) Choose a system of parameters $x_1, \ldots, x_k \in m$ for R. Since $m = \mathrm{Rad}\,(x_1, \ldots, x_k)R$, S/mS has the same dimension, say d, as as $S/(x_1, \ldots, x_k)S$, and if we choose elements $y_1, \ldots, y_d$ in S whose images in $S/(x_1, \ldots, x_k)S$ form a system of parameters, we have that $n \subseteq \mathrm{Rad}\,(x_1, \ldots, x_k, y_1, \ldots, y_d)S$. Thus, $\dim(S) \le k + d$,and $k \ge \dim(S) - d$. $\square$

(b) Let m be the maximal ideal of R corresponding to y. Then $\dim f^{-1}(y) = \dim(S/mS) = \dim(S/mS)_n$ for some maximal ideal $n \supseteq mS$, which equals $\dim(S_n/mS_n) \ge \dim(S_n) - \dim(R_m)$ (by (a)) $= \dim(S) - \dim(R) = \dim(X) - \dim(Y)$.

4. The "if" part is true: by the lying over theorem the map is surjective, and the scheme-theoreric fibers are module-finite over a field: these rings are zero-dimensional Noetherian rings, and therefore are Artin and have only finitely many primes, all of which are maximal. The "only if" part is false: see the solution of the next problem.

5. Since f is bijective the fibers are 0-dimensional, and $K(X)$ is algebraic over $K(Y)$. Since there is one point in every fiber, and the extension is separable, $[K(X) : K(Y)] = 1$, i.e., $K(X) = K(Y)$. But the bijectivity does not imply that S is contained in the the integral closure T of R in its fraction field. We use the ring $R = K[t^2 - 1,\, t^3 - t]$ from Problem **6.** to give a counter-example. The integral closure of R in its fraction field is $K = R[t]$. By the results of **6**, R is the coordinate ring of $Y = \mathcal{V}(y^2 - x^2 - x^3) \subseteq \mathbb{A}^2_K$. The map $R \subseteq K[t,\, 1/(t+1)$ induces a bijection of $X = A^1_K - \{-1\}$ with Y. Note that there is at least one point of $\mathbb{A}^1_K$ lying over a given point of Y, since $K[Y] \subseteq K[t]$ is module-finite. Over points (x, y) of Y where $x \neq 0$, one can recover t as y/x. Over the point where $x = 0$ (y must be 0 as well), one must have $t^2 - 1 = 0$, and so one gets only the point corresponding to $t = 1$ in X. While $K(Y) = K(X)$, it is not true that $K[X]$ is contained in the integral closure $T = K[t]$ of $K[Y]$ in $K(Y)$.

6. The defining equation $y^2 - x^2 - x^3$ of Y is irreducible, since $x^2 + x^3 = x^2(1 + x)$ is not a perfect square in $K(x)$. Hence, the quotient ring is a 1-dimensional domain R. Under the K-algebra map $K[x, y] \to K[t]$ such that $x \mapsto t^2 - 1$ and $y \mapsto t^3 - t$, the element $y^2 - x^2 - x^3 \mapsto (t^3 - t)^2 - (t^2 - 1)^2 - (t^2 - 1)^3 = (t^2 - 1)^2\big(t^2 - 1 - (t^2 - 1)\big) = 0$, and so there is a K-algebra map $h : R \twoheadrightarrow K[t^2 - 1, t^3 - t] \subseteq K[t]$. Since this is a surjection of 1-dimensional domains, $\mathrm{Ker}\,(h) = (0)$, and h is an isomorphism. Since $t = (t^3 - t)/(t^2 - 1)$ is in the fraction field and satisfies, e.g., $z^3 - z - (t^3 - t) = 0$, t is in the integral closure and, since $K[t]$ is normal, $T \cong K[t]$. Note that $t \notin R$, since no element of R has degree one. We may identify R with $K[t^2 - 1, t^3 - t]$ and T with $K[t]$. Given any polynomial $F \in K[t]$ of degree $N > 1$, we can write $N = 3a + 2b$ for integers $a,\, b \in \mathbb{N}$, and then $\deg\big(F - (t^3 - t)^a(t^2 - 1)^b\big) < \deg(F)$. By induction on the degree, $T = K[t] = R + Kt$, and so $di_K(T/R) = 1$. Hence, $g \in R$ is in $\mathfrak{A}$ iff $gt \in R$. Then $t^2 - 1 \in \mathfrak{A}$, since $t^3 - t \in R$, and $t^3 - t \in \mathfrak{A}$, since $(t^3 - t)t = (t^2)^2 - (t^2) \in R$, since $t^2 = (t^2 - 1) + 1$. Thus, $\mathfrak{A} \supseteq (t^3 - t,\, t^2 - 1)R = (x,\, y)R$, which is maximal. Hence, $\mathfrak{A} = (t^3 - t,\, t^2 - 1)R$.

EXTRA CREDIT 2. The matrix $\big(I_n \,|\, v\big)$ where I_n is an $n \times n$ identity matrix, $|$ indicates concatenation, and the column v has entries $c_1, \ldots, c_n$ maps to $(-1)^{n-1}(c_1, -c_2, \ldots, \pm c_n, 1)$. Multiply the first row by $c \neq 0$ to get any vector with last entry $\neq 0$ in the image of f. By symmetry, all of $\mathbb{A}^{n+1}_K - \{0\}$ is in the image of the matrices of rank n. All matrices of rank $< n$ map to 0. Let $\alpha \in G = \mathrm{GL}(n+1, K)$ act on X by right multiplication, i.e., $\mu \mapsto \mu\alpha$, and on Y (viewed as row vectors) via $v \mapsto v\det(\alpha)$. Then $f(\mu\alpha) = f(\mu)\det(\alpha)$, since we may identify $f(\mu)$ with $\wedge^n(\mu)$. Any two $n \times (n+1)$ matrices of rank n are in the same orbit of G (bring each to reduced column echelon form, which will be $\big(I_n \,|\, 0\big)$ for both, by elementary column operations — these are given by right multiplication by elements of G). Hence, all fibers over $y \in \mathbb{A}^{n+1}_K - \{0\}$ are isomorphic. Since $\dim f^{-1}(Y) = (n^2 + n) - (n + 1) = n^2 - 1$ for y in an open set $\neq \emptyset$, $\dim f^{-1}(y) = n^2 - 1$ for all $y \in Y - \{0\}$. Let $V = f^{-1}(0) = \{\mu : \text{rank }(\mu) < n\}$, and rank $(\mu) < n$ iff $\mathrm{Im}(\mu) \subseteq W \subseteq K^n$ with $\dim_K(W) = n-1$. Thus, rank $(\mu) < n$ iff μ factors $K_{n+1} \to K_{n-1} \to K_n$, i.e., $\mu = \beta\gamma$ where γ is $n \times (n-1)$ and β is $(n-1) \times (n+1)$. This gives a surjection $\mathbb{A}^{n(n-1)}_K \times A^{(n+1)(n-1)}_K \twoheadrightarrow V$, so that V is irreducible, and $\dim(V) = \dim(U)$, where $U \subseteq V$ is the open set such that the first $n-1$ rows are independent. For matrices in U, the last row is uniquely a linear combination of the first $n-1$ rows, which gives a bijection $\mathbb{A}^{(n+1)(n-1)}_K \times \mathbb{A}^{n-1}_K \twoheadrightarrow U$ such that (θ, w) maps to the matrix whose first $n-1$ rows give θ and whose last row is $w\theta$. Hence, $\dim f^{-1}(0) = (n+1)(n-1) + (n-1) = (n+2)(n-1)$.