Math 615, Winter 2017 **Problem Set #2: Solutions**

1. We may replace n by a large power of 2, say $2^k \geq n$, and assume that $J^{2^k} = 0$ for some k. We use induction on k. The case $k = 1$ follows from the definition. For the inductive step, we have, if $J^{2^{k+1}} = 0$, that there is a (resp., at most one, resp., a unique) lifting to $T_1 = T/J^{2^k}$, since $(JT_1)^{2^k} = 0$ (by the induction hypothesis). But then there is a (resp., at most one, resp., a unique) lifting to T, since $(J^{2^k})^2 = 0$ in T.

2. (a) Because all exponents on all X_i are divisible by p, they contribute 0 to the partials, and the Jacobian matrix $\mathcal{J}$ equals that of their linear forms, which consists of their coefficients. Since the linear forms are linearly independent, so are the columns of $\mathcal{J}$ (a matrix of scalars over K), and so $\det(\mathcal{J}) \in K - \{0\}$. Thus, the map is étale. $\square$

(b) If $h = 0$ the quotient, R, is étale. If $h = 1$, the quotient $R[x]/(c)$ is 0, hence, étale if $c \neq 0$ and smooth but not étale if $c = 1$. Assume $h \geq 2$. We need the invertibility of the derivative $hx^{h-1} - 1$ modulo $x^h - x + c$. Modulo $hx^{h-1} - 1$, $x^h - x + c$ becomes $(1/h)x - x + c$. We have an étale extension unless $x = c/(1 - 1/h)$ satisfies $hx^{h-1} - 1 = 0$. Thus, the condition is that $hc^{h-1}/(1 - 1/h)^{h-1} - 1 \neq 0$ or $c^{h-1} \neq (1 - h)^{h-1}/h^h$.

3. (a) By base change, $S_1 \otimes_R S_2$ has the property needed (smooth, unramifeid, or étale) over S_2, and S_2 has the property needed over R. Thus, we may compose. $\square$

(b) Write d_i for the universal R-derivation $d_{S_i/R} : S_i \to \Omega_{S_i/R}$. Define an R-linear map $D : S_1 \otimes_R S_2 \to S_2 \otimes_R \Omega_{S_1/R} \oplus S_1 \otimes_R \Omega_{S_2/R}$, which we call Ω, using the map induced by the R-bilinear map that sends $(s_1,\, s_2) \mapsto s_2 \otimes d_1(s_1) + s_1 \otimes d_2(s_2)$, which will then be the value of $D(s_1 \otimes s_2)$. Then D is R-linear. Since the elements of the form $s_1 \otimes s_2$ span, to check that D is a derivation it suffices to check this for $(s_1 \otimes s_2)(s_1' \otimes s_2')$, which is straightforward. To complete the proof, it suffices to show that D is universal. Let $d : S_1 \otimes S_2 \to W$ be any other R-derivation. Composing with the map $S_1 \to S_1 \otimes S_2$ gives an R-derivation $S_1 \to W$ that factors uniquely $S_1 \xrightarrow{d_1} \Omega_{S_1/R} \xrightarrow{L_1} W$. We may construct L_2 simiilarly, and $\mathrm{id}_{S_2} \otimes L_1 \oplus \mathrm{id}_{S_1} \otimes L_2$ gives an $S_1 \otimes_R S_2$ linear map $L : \Omega \to W$ such that $L \circ D = d$. It is clear that L is the only map that can work, since $d(s_1 \otimes s_2) = s_2 d(s_1) + s_1 d(s_2)$ must be the image of $s_2 \otimes d_1(s_1) \oplus s_1 d_2(s_2)$.

4. Let J be the annihilator of the finitely generated ideal $I = (h_1, \, \ldots \, , h_k)R$. If $I + J$ is proper, it is contained in a maximal ideal m. Then $IR_m \subseteq mR_m$ and $IR_m = (IR_m)^2$. By Nakayama's Lemma, $I = 0$ in R_m. For each generator h_i of I, let $g_i \in R - m$ kill h_i, and so $g = \prod_i g_i \in R - m$ kills I, contradicting that $J \subseteq m$. Hence, $I + J = R$, and we can choose $e \in I$ with $1 - e \in J$,, so that $(1 - e)h = 0$ for all $h \in I$, and so $he = h$. Then $e \in I$ is idempotent and $I = eR$. When S is finitely generated by $s_1, \, \ldots \, , s_n$, $I = \mathrm{Ker}\,(S \otimes_R S \to S)$ is finitely generated by the $s_i \otimes 1 - 1 \otimes s_i$, and so $I = I^2$ (class criterion for unramifed) iff I is generated by an idempotent.

5. Suppose that we have an R-algebra map $S_Q \to T/J$ with $J^2 = 0$. Since S is formally smooth, the map $S \to S_Q \to T/J$ lifts uniquely to a map $S \to T$. Since the image of $S - Q$ is invertible in T/J and $J^2 = 0$, it is invertible in T. Hence, this induces a lifting $S_Q \to T$, and since its values on S are uniquely determined, the same is true for its values on S_Q. (Alternatively, but not from directly from the definition, R to S is formally étale, so are all localizations $R \to S_Q$, since each $S \to S_Q$ is formally étale, and we may compose.) We prove the converse when S is finitely presented. In that case S is étale iff it is flat and unramified over R. Since S_Q is formally unramified for all Q, every $(\Omega_{S/R})_Q = 0$, and so $\Omega_{S/R} = 0$. By the Jacobian criterion for smoothness, S is smooth near Q if and only if S_Q is formally smooth, and if S is smooth near Q we will have that S_b is flat over R_a and hence over R, for $a \notin P$ with invertible image in S_b with $b \notin Q$. This implies that S_Q is flat over R. Since whenever S_Q is formally étale it is formally smooth, it follows that every S_Q is flat over R, and so S is flat over R.

In considering whether a map from R to (S, Q) has the property of being formally smooth, or formally unramified, or formally étale, it does not matter whether we consider S as an algebra over R or R_P because the R-homomorphisms from S to any R-algebra T are the same as the R_P-homomorphisms, since the elements of $R - P$ map to units in (S, Q) and hence to units in T/J and so to units in T (since J is nilpotent).

6. Note that the equation $y^{2p} + uy^p - v$ is irreducible over $k[y, u, v]$ (the quotient is $k[y, u]$). Hence, it is irreducible and generates a prime ideal over the localization $K[y]$. It is not a separable equation since both of the exponents on y are divisible by p. For the final statement, that L contains an element w of $K^{1/p}$ not in K. Then $[K[w] : K] = p$, and so $[L : K[w]] = 2$. It follows that y satisfies a monic quadratic equation over $K[w]$. But if we enlarge $K[w]$ to all of $K^{1/p}$ we know the quadratic equation that y satisfies: $y^2 + u^{1/p}y - v^{1/p} = 0$, which is clearly irreducible over $K^{1/p} = k(u^{1/p}, v^{1/p})$. This quadratic is unique, so we must have that $u^{1/p}$, $v^{1/p}$ are both in $K[w]$, a contradiction, since adjoining both produces an extension of K of degree p^2. The exclusion of $p = 2$ is not needed.

EXTRA CREDIT 3. As in class, the lifting problem from T/I to T when $I^2 = 0$ comes down to solving equations $F_j(X_1 + \delta_1, \ldots X_n + \delta_n) = 0$ for the δ_i in I, where one replaces the coefficients by their images in T and the X_i by arbitrarily chosen liftings of their images in T/I to T. The image of $F_j(x_1, \ldots, x_n)$ is then $v_j \in I$. By Taylor's formula, the image of $F_j(X_1 + \delta_1, \ldots X_n + \delta_n) = 0$ is $v + \sum_{i=1}^{n} (\partial F_j/\partial X_j)\delta_i$. In matrix form, this equation is $-(v_1, \ldots, v_n) = (\delta_1, \ldots, \delta_n)\big(\partial F_j/\partial X_i\big)$ Here, siince the $\delta_i \in I$, which is a T/I module, the action of the matrix $\big(\partial F_j/\partial X_i\big)$ is determined by its image matrix $\mathcal{J}$ over T/I, and so this action is determined by the image $\mathcal{J}$ over S. Hence, it is sufficient for the map to be unramified (resp., smooth) if the action of $\mathcal{J}$ by *right* multiplication is injective (resp., surjective) for all modules M. Thus, one needs to show that the action of a matrix on the right is injective for all modules iff its action on the left is surjective for all modules, and conversely. See the solution to **Extra Credit 4**.

EXTRA CREDIT 4. If $m \leq n$ the map is injective for all n if and only ithe $m \times m$ minors of the matrix generate the unit ideal. if $m \geq n$, the map is surjective for all M if and only if the $n \times n$ minors generate the unit ideal. These conditions are necessary, because if the minors do not generate the unit ideal, one can choose a maximal ideal P that contains them all, and take $M = R/P$. The condition for injectivity (resp., surjectivity) is that the matrix have rank m (resp., n). The conditions are sufficient because if there is a nonzero kernel (resp., a cokernel) there will still be one after localizing at a suitable maximal ideal. But then, as in class, one may bring the matrix to the form where there are all ones on the main diagonal and zeroes elsewhere by elementary row and column operations, and the issues are unaffected. The result needed for **Extra Credit 3.** is now immediate by considering the matrix and its transpose. There are many other correct characterizations. A matrix gives a map that remains surjective after tensoring with any module if and only if it is sujrective, by the right exactness of tensor. It is also true that a matrix gives a map that remains injective if and only if the cokernel is locally free (this follows from the characterization of freeness of the cokernel in the quasilocal case given in class). Moreover, for a finitely presented module, being flat, being projective, and being locally free are equivalent properties. $\square$

EXTRA CREDIT JC. This is an open question for all $n \geq 2$.