Math 615, Fall 2017 **Problem Set #3**

Due: Wednesday, March 15

1. Prove that a direct limit of formally unramified (respectively, formally étale) R-algebras is formally unramified (respectively, formally étale).

2. Let V_p denote the completion of the discrete valuation ring $\mathbb{Z}_P$, where P is generated by the positive prime integer p. For which integers $n \geq 2$ and primes p not dividing $n+1$ does the equation $x^n + x + p - 2 = 0$ have a unique solution $r \in V_p$ such that $r \equiv 1$ modulo (p)? Prove your answer.

3. Show that a direct limit of Henselian quasilocal rings in which the homomorphisms are local is Henselian.

4. Let K be a field of characteristic 0, and $X = X_0$ an indeterminate. Determine whether $\bigcup_{n=1}^{\infty} K[X_n]$, where $X_n = X^{1/2^n}$ in such a way that $X_{n+1}^2 = X_n$ for all n, is formally smooth over K. Prove your answer.

5. Let (R, m, K) be a Henselian ring. Let $Z_1, \ldots, Z_n$ be indeterminates over R: the subscripts should be read modulo n. Let $u_1, \ldots, u_n \in m$ and let $r_1, \ldots, r_n \in R$. Let $h_1, \ldots, h_n$ be integers all of which are ≥ 2.

Show that the n simultaneous equations

$$\begin{aligned} u_1 Z_1^{h_1} + Z_2 &= r_1 \\ &\cdots \\ u_i Z_i^{h_i} + Z_{i+1} &= r_i \\ &\cdots \\ u_n Z_n^{h_n} + Z_1 &= r_n \end{aligned}$$

have a solution in R.

6. Let $A = K[[x, y]]$ be the formal power series ring in two variables over a field K. (You may assume that A is a UFD.) Let $P = xA$, which is a prime ideal. Prove that the local ring $R = A_P$ is *not* Henselian by showing that there is a monic polynomial $F = F(Z)$ in one indeterminate over R such that when F is considered modulo PR it has simple roots, but such that F has no root in R.

EXTRA CREDIT 5. Let $K_1, \ldots, K_n, \ldots$ be fields, and let $R = \prod_{n=1}^{\infty} K_n$ be their product.

(a) Show that every element of R is the product of a unit and an idempotent, and show that R is a zero-dimensional ring, i.e., that every prime ideal is maximal.

(b) Suppose all of the K_n are K-algebras, where K is a field, and assume as well either that (1) infinitely many K_n are infinite fields or (2) there is an infinite set of K_n of finite cardinality such that the cardinals of these K_n are not bounded. ((1) is immediate if K is infinite.) Show that R contains an element that is transcendental over K.

(c) Suppose also that K has characteristic 0 (e.g., this holds if $K = K_n = \mathbb{Q}$ for all n). Show that R is *not* formally unramified over K.

EXTRA CREDIT 6. Must a formally étale algebra S over a field K be reduced? Prove this, or give a counterexample.