Math 615, Winter 2017 **Problem Set #3 Solutions**

1. Let $\{S_\lambda\}_\lambda$ be the direct limit system of R-algebras, where $h_{\lambda,\mu} : S_\lambda \to S_\mu$ if $\lambda \le \mu$ and $h_\lambda : S_\lambda \to S$, where S is the direct limit. In the unramifiied case, given two different liftings $f,\, g : S \to T$, there is some element $s \in S$ such that $f(s) \neq g(s)$. But s must be the image of an element $s_\lambda \in S_\lambda$ for some choice of λ, and then the two composite maps $S_\lambda \to S \to T$ obtained from composing f, g with h_λ disagree on s_λ, a contradiction. Alternatively, $\Omega_{S/R}$ is the direct limit of the modules $\Omega_{S_\lambda/R}$, all of which are 0. For the étale case, given a map $f : S \to T/I$ we have maps $f \circ h_\lambda : S_\lambda \to S \to T/I$. If there is a unique lifting to a map $\widetilde{f}_\lambda : S_\lambda \to T$ for all λ, it follows that for all $\lambda \le \mu$ the map $\widetilde{f}_\mu \circ h_{\lambda,\mu}$ gives the lifting $S_\lambda \to T$. By the definition of direct limit, we get a lifting $S \to T$. $\square$

2. This is true for all primes p not dividing $n+1$, since this complete local ring is Henselian and 1 is a simple root of the polynomial mod pV (since the value of the derivative $nx^{n-1}+1$ at 1 is $n+1$, which is not 0 mod pV.

3. It suffices to show that the direct limit ring $(R,\, m)$ has no proper pointed étale extension. Such an extension must be standard, and so have the form $S = \big((R[X]/(F)\big)_{\mathcal{Q}}$, where F is monic, $\mathcal{Q} = (m, X-r)$ for some $r \in R$ such that the image o r in $R/m = K$ is a simple root of F, i.e., $F'(r)$ is a unit of R. The coefficients of F and the element r must be images of elements from some ring $(R_\lambda,\, m_\lambda,\, K_\lambda)$ in the direct limit system: this is true for each of them individually, and so must be true for any sufficiently large λ. Suppose that F_λ is a monic polynomial over R_λ that maps to F, and that r_λ is an element of R_λ that maps to $r \in R$. Then $F_\lambda(r_\lambda)$ maps to $F(r) \in m$, and we must have that $F_\lambda(r_\lambda) \in m_\lambda$. Moreover, $F'_\lambda(r_\lambda)$ maps to $F'(r)$, which is a unit of R, and this implies that $F'_\lambda(r_\lambda)$ is a unit of R_λ. Thus, if $\mathcal{Q}_\lambda = (m_\lambda, X - r_\lambda)$, we have that $\big(R_\lambda[X]/(F_\lambda)\big)_{\mathcal{Q}_\lambda}$ is a pointed étale extension of the Henselian ring R_λ, and is therefore isomorphic to R_λ. Under the isomorphism, the image of X is identified with an element $r'_\lambda \in R_\lambda$, where $r'_\lambda \equiv r_\lambda$ mod m_λ. Let r' be the image of r'_λ in R: we also have that $r' \equiv r$ mod m. Since we have local maps $R_\lambda \cong \big(R_\lambda[X]/(F_\lambda)\big)_{\mathcal{Q}_\lambda} \to \big((R[X]/(F)\big)_{\mathcal{Q}}$, the image of X in the rightmost ring must $\mathcal{Q}$ may be identified iwth m. Thus, $R \to S$ is an isomorphism. $\square$

4. Let $Y_1,\, \ldots\, , Y_n$ for $n \ge 1$ be new indeterminates, and form the ring $T = K[Y_1,\, \ldots\, , Y_n]/J^2$, where J is the ideal generated by all the $Y_{n+1}^2 - Y_n$ for $n \ge 1$. Let $I = J/J^2$. There is an obvious K-algebra map $\theta : R \cong T/I$ that sends $X_n \mapsto Y_n$ for all $n \ge 1$. We shall show that R is *not* smooth over K by showing θ can't be lifted to a map $R \to T$. Let $Z_n = Y_{n+1}^2 - Y_n$. Let y_n, z_n be the respective images of Y_n, Z_n for all n.

Let B denote the ring $K[Z_1,\, \ldots\, , Z_n,\, \ldots]/(Z_1,\, \ldots\, , Z_n,\, \ldots)^2$. This is a K-vector space with basis $1,\, z_1,\, \ldots\, , z_n,\, \ldots$. We may think of T as an ascending union of rings B_n where $B_0 = B$, $B_1 = B[Y_1]$, and for all n, $B_{n+1} = B[Y_{n+1}]/(Y_{n+1}^2 - y_n - z_n)$, which is free on the basis $1, y_{n+1}$ over B_n. It follows that $T \cong \bigcup_{n=0}^\infty B_n$ is free over $B_1 = B[y_1]$ on the basis consisting of all monomials in the y_n, $n \ge 2$, that are linear in each y_j. Note that $I = J/J^2$ is the ideal of T generated by the z_k. Since $I^2 = 0$, I is a T/I-module, and, in fact it is free over $T/I \cong R$ on the basis $z_1,\, \ldots\, , z_n,\, \ldots$. It will be convenient to write elements of I as R-linear combinations of the z_j.

The problem of lifting the map $R \cong T/I$ to a K-algebra map $R \to T$ is equivalent to finding elements $\delta_n \in I$ such that $y_n + \delta_n$ can serve as the image of $x_n \in R$, which means that the equations $(y_{n+1} + \delta_{n+1})^2 = y_n + \delta_n$ must be satisfied for all $n \geq 1$. Since δ_{n+1}^2 will be zero for $\delta_{n+1} \in I$, this means that $y_{n+1}^2 - y_n + 2\delta_{n+1}y_{n+1} = \delta_{n+1}$ for all $n \geq 1$, and so $\delta_n = z_n + 2y_{n+1}\delta_{n+1}$, $n \geq 1$. Thus, $\delta_1 = z_1 + 2y_2\delta_2 = z_1 + 2y_2(z_2 + 2y_3\delta_3) = z_1 + 2y_2z_2 + 4y_2y_3\delta_3$, and continuing in this way one obtains by a straightforward induction on n that

$$(*) \quad \delta_1 = z_1 + 2y_2z_2 + 4y_2y_3z_3 + 8y_2y_3y_4z_4 + \cdots + 2^{n-1}(y_2y_3\cdots y_n)z_n + 2^n(y_2y_3\cdots y_{n+1})\delta_{n+1}$$

for all $n \geq 2$.

But δ_1 has a unique representation as an R-linear combination of the z_i, and only finitely many z_i can have a nonzero coefficient. We can therefore obtain a contradiction by showing that the coefficient of every z_k is nonzero. To see this, choose $n = k$, let h be the coefficient of z_k occurring in δ_{k+1}, and let $\overline{h}$ be the image of h in R. From $(*)$ with $n = k$, we see that the coefficient of z_k in δ_1 is the image of $2^{k-1}(y_2y_3\cdots y_k) + 2^k(y_2y_3\cdots y_k)y_{k+1}h$ in R, which is $2^{k-1}x_2x_3\cdots x_k(1 + 2x_{k+1}\overline{h}$. This cannot vanish since all of the x_j are nonzero, and x_{k+1} is not a unit in the domain R. Many other solutions are possible. $\square$

5. Let $S = R[Z_1, \ldots, Z_n](u_iZ_i^{h_i} + Z_{i+1} - r_i : 1 \leq i \leq n)_{\mathcal{Q}}$ where $\mathcal{Q}$ is the maximal ideal generated by m and the elements $Z_{i+1} - r_i$, $1 \leq i \leq n$. It will suffice to show that S is a pointed étale extension of R, since that implies $R \to S$ is an isomorphism, and the images of the Z_i in S will correspond to elements of R that necessarily satisfy the given system of equations. Because the $u_i \in m$, the image of the Jacobian matrix modulo m is the same as if the generators of the ideal were $Z_{i+1} - r_i$, and the matrix of partial derivatives is clearly a permutation matrix in this case, and so is invertible, with determinant ± 1. This shows that the extension is étale. Since the residue class field modulo $\mathcal{Q}$ is still $K = R/m$, this is a pointed étale extension (the Z_{i+1} map to the images of the respective r_i in K). $\square$

6. If the characteristic of K is not 2, we may use $Z^2 - 1 - (1/y^2)x$ to give an example. Mod $PR_P = xR_P$, this has the two simple roots, ± 1. We claim that it has no roots even in the fraction field of $K[[x, y]]$. If it did, the element $u = 1 + (1/y^2)x$ would be a perfect square in the fraction field, and, hence, so would $y^2u = y^2 + x$. Since $K[[x, y]]$ is a UFD, it is normal, and this would mean that a fractional square root of $y^2 + x$ would be in the ring $K[[x, y]]$. Such a root cannot be a unit, since $y^2 + x$ is not a unit, and cannot be in the maximal ideal m, since $y^2 + x \notin m^2$. In characteristic 2, one may given a very similar argument with $Z^2 + Z - (1/y^2)x$, which has the roots $0, 1$ mod PR_P. If there were a root r in the ring, then $v = yr$ would satisfy that $v^2 + yv - x = 0$. Since this is monic in v, we would then have $v \in R$. This shows that $v^2 \in (x, y) = m$ and so $v \in m$. But then $x \in y + m^2$, which is false. There are many similar examples. $\square$

EXTRA CREDIT 5. (a) Consider an element r of the product. Let e denote the element of the product which is 1 in each coordinate where r is nonzero and 0 in the other coordinates. Let u be the element which agrees with r in each coordinate where r is nonzero and is 1 elsewhere. The e is idempotent, u is a unit, and $r = ue$. Note that $r^2 = u^2e^2 = u^2e$ and that $r = u^{-1}r^2$. Let P be a prime ideal of R, and $r \notin P$. With u as described, $r - u^{-1}r^2 = 0 \in P$, i.e., $r(1 - u^{-1}r) \in P$. Since $r \notin P$, $1 - u^{-1}r \in P$, which shows that the image of u^{-1} is an inverse for r in R/P. Thus, R/P is a field.

(b) If any of the fields is transcendental over K, we may choose a transcendental element in the coordinate corresponding to that field and arbitary elements elsewhere. Thus, we may assume that all of the fields are algebraic over K, and we may identify them with subfields of an algebraic closure L for K. We choose a countable subsequence of the fields and construct an element of the product as follows. Pick any of the fields to be K_{n_1} and pick $c_{n_1} \in K_1$ arbitrarily. Now suppose that $c_{n_1} \in K_{n_1}, \ldots, c_{n_h} \in K_{n_h}$ have been chosen. Choose $K_{n_{h+1}}$ such that its cardinality is greater than h, and so contains an element c_{h+1} that is distinct from $c_{n_1}, \ldots, c_{n_h}$. Let r in the product be such that its coordinate corresponding to K_{n_i} is c_{n_i}. All other coordinates may be chosen arbitrarily, e.g., we may take them all to be zero. Let f be any non-trivial polynomial over K. Then $f(r) \neq 0$, for there infinitely many distinct c_{n_i}, and so they cannot all be roots of f. $\square$

(c) If R is formally ramified over K, so is every homomorphic image. Choose an element r of R that is transcendental over K. There is a prime ideal P of R disjoint from the multiplicative system $K[r] - \{0\}$. Then $F = R/P$ is also formally unramified over K. This is a field transcendental over K, since P is maximal (even if we did not know this, we could localize at $R - P$ to get a field that is formally unramified over K). Choose a transcendence basis $\{x_\lambda\}_\lambda$ for this field: it is not empty. By class results, if $L = K(x_\lambda : \lambda)$, $\Omega_{L/K}$ has the dx_λ as a K-vector space basis. Since K has characteristic 0, F is separable algebraic over $K(x_\lambda : \lambda)$. By another class theorem, $\Omega_{F/K}$ has the dx_λ as an F-vector space basis over F.

EXTRA CREDIT 6. We first show that if $R \to S$ is a homomorphism of rings of characteristic p such that Frobenius is surjective on R and is an automorphism of S, then S is formally étale over R. It is certainly formally unramified: since every element s of S has the form u^p, we have $ds = pu^{p-1}du = 0$. Suppose we have an R-algebra map $S \to T/I$, where $I^2 = 0$, and we wish to lift it to a map $S \to T$. For each $s \in S$, choose u_s such $u_s^p = s$ (u_s is unique). Let t_s lift the image of u_s in T/I to T. We define the lifted homomorphism by letting it take s to t_s^p. Note that if we change the choice of t_s by adding an element of i, we have $(t_s + i)^p = t_s^p + i^p = t_s^p$, so this is independent of choices. It is then easy to check that it preserves sums and products, since the sum (or product) of lifts of two elements gives a lift of their sum (or product). Moreover, the map is R-linear, for if $r \in R$ and $a^p = r$, we have that $(au)^p = rs$, and at is a lift of the image of au to T. Hence, $a^p t^p = rt^p$ will be the image of rs.
Let K be a perfect field of characteristic $p > 0$. Form a directed union of polynomial rings R_n as follows. Let $R_1 = K[X_1]$. Let $R_2 = K[X_{11}, \cdots, X_{12}]$ and note that we may inject $R_1 \hookrightarrow R_2$ by the K-algebra map sending $X_1 \mapsto X_{11}X_{12}$. In general, let R_n be the polynomial ring in 2^n variables over K indexed by all strings of length n consisting of the n symbols, each of which is 1 or 2, and the K-algebra map $R_n \hookrightarrow R_{n+1}$ sends $X_\sigma \mapsto X_{\sigma\,1}X_{\sigma\,2}$ for each string σ of length n. In the directed union T_0, $m = m^2$, where m is generated by all the X_σ. Now let T be the ring obtained by adjoining p^e h roots of all elements of T (say from an algebraic closure of its fraction field). Since the Frobenius endomorphisms F_K and F_T are both automorphisms, it follows from the result of the first paragraph that T is formally étale and therefore formally smooth over K. Since $m = m^2$, if $I = mT$ we have that $I = I^2$. Hence, the map $T/I^2 \twoheadrightarrow T/I$ is an isomorphism: consequently, it splits as a map of K-algebras. This implies that T/I is formally smooth over K, and it is

formally unramified since T is. Thus, T/I is formally étale over K. In this ring, the image of $z = X_1^{1/p}$ is nilpotent, since its pth power is in m. It will suffice to show that z is not 0. If A is a domain, let $A^{1/q}$ denote the result of adjoining all qth roots to A, where $q = p^e$. Then T is the union of the rings $R_n^{1/q}$ as n and q vary. If z were 0 in T, then in some $R_n^{1/q}$ we would have that $X_1^{1/q} \in n_n R_n 1/q$, where m_n is generated by the X_σ for σ of length n. Raising to qth powers, we would have that $X_1 \in (X_\sigma^q : \sigma)R[x_\sigma : \sigma]$ where the σ runs through strings of 1s and 2s of length n, and $X_1 = \prod_\sigma X_\sigma$ in R_n. This is clearly false. $\square$