Math 615, Fall 2017 **Problem Set #4**
Due: Monday, April 3

1. (a) Show that if S is smooth over a quasilocal ring $(R,\, P)$, then R is a direct limit of local Noetherian subrings $(R_\lambda,\, P_\lambda)$ and local maps, and that one can choose one of these, say R_0, and a smooth extension S_0 of R_0 such that $S = S_0 \otimes_{R_0} R$. Hence, S is the direct limit of the rings $S_0 \otimes_{R_0} R_\lambda$ for $\lambda \geq \lambda_0$.

(b) Let $(R,\, P) \to (S,\, Q)$ be a flat local homomorphsim of quasilocal rings such that $(R,\, P)$ is reduced. Suppose that R has only finitely many minimal primes (which holds, for example if R is Noetherian), and that the fiber over every minimal prime is reduced. Show that S is reduced.

(c) Show that if R is reduced and S is essentially smooth over R, then S is reduced.

(d) Show that if $(R,\, P)$ is quasilocal and reduced, then its Henselization is reduced.

2. Let R be the localization of a domain of Krull dimension one finitely generated over the complex numbers $\mathbb{C}$. Show by example that the Henselization of R need not be a domain.

3. Let (R, P, K) be a quasilocal ring and let M be an $s \times s$ matrix over R that is congruent to the identity matrix mod P. Prove that if n is a positive integer not divisible by the characteristic of K, then M has an nth root over a pointed étale extension of R.

4. Let K be a field and let R be a reduced K-algebra. Prove that R is separable over K if and only if $\mathrm{frac}(R/P)$ is separable over K for every minimal prime P of R.

5. Let (R, P, K) be a quasilocal ring and let N be the ideal of all nilpotent elements of R. Suppose that R/N is Henselian. Prove or disprove that R must be Henselian.

6. Let (R, m, K) be an approximation ring. Show that if R has the property of being reduced, or of being a domain, then so does its m-adic completion $\widehat{R}$.

EXTRA CREDIT 7. Let R be Noetherian and formally smooth over a perfect field K. Prove that R is regular. [Suggestion: reduce to the local case, (R, P). You may assume that since K is perfect, it is contained in a coefficient field L for R/P^2 (which is complete). (This is clear in char. 0. In char. $p > 0$, any perfect field $\kappa \subseteq$ is contained in *every* coefficient field. Cf. the Theorem on p. 12 of the supplement on *The structure theory of complete local rings*: $\kappa = \kappa^{p^n} \subseteq R^{p^n} \subseteq R_n$.) Let $x_1, \, \dots \, , x_d$ be a minimal set of generators of P. We have $R \twoheadrightarrow R/P^2 \cong L[X_1, \, \dots \, , X_d]/m^2$, where $\underline{X} = X_1, \, \dots \, , X_d$ are indeterminates and $m = (\underline{X})$. Use that R is formally smooth over K to show this lifts to a map $R \twoheadrightarrow L[\underline{X}]/m^{n+1}$ for all n, whence $\ell(R/P^{n+1}) \geq \ell(L[\underline{X}]/m^{n+1})$. Conclude that $\dim(R) \geq d$.]

EXTRA CREDIT 8. Let (R, m, K) be an approximation ring. Show that if R is normal, then so is its m-adic completion $\widehat{R}$.