Math 615, Winter 2017 **Problem Set #4 Solutions**

1. (a) For every subring A of R finitely generated over the prime ring $\mathrm{Im}\,(\mathbb{Z} \to R)$, we may consider the localization $(B,\, P_B)$ of B at the contraction of P to B. Clearly, $(R,\, P)$ is the directed union of the rings $B,\, P_B)$, and the inclusion maps are local. Choose a finite presentation $S = T/I$ for S, where $T = R[X_1,\, \ldots\,, X_n]$ is a polynomial ring over R, and $I = (F_1,\, \ldots\,, F_m)$. Choose B_0 sufficiently large that $F_1,\, \ldots\,, F_m \in B_0[X_1,\, \ldots\,, X_n]$, and for any $B \supseteq B_0$, let $T_B = R[X_1,\, \ldots\,, X_n]$, let $I_B = (F_1,\, \ldots\,, F_m)T_B$, and let $S_B = T_B/I_B^2$. Since T/I is smooth over R, $T/I^2 \twoheadrightarrow T/I$ splits over R. Suppose that the image of X_j under $T \to T/I \to T/I^2$ is represented by $X_j + \sum_{h=1}^{m} G_{jh}F_h$ (it must have this form). Moreover, $(*)$ for every i, $F_i(X_1 + \sum_{h=1}^{m} G_{1h}F_h, \ldots\,, X_n + \sum_{h=1}^{m} G_{nh}F_h)$ must be in I^2, and so must be expressible as $\sum_{j,k} H_{ijk}F_jF_k$. Choose $B \supseteq B_0$ large enough to contain all the coefficients of the G_{jh} and the H_{ijk}. Then we can define $T_B/I_B \to T_B/I_B^2$ so that under $T_B \to T_B/I_B \to T_B/I_B^2$ the image of X_j is represented by $X_j + \sum_{h=1}^{m} G_{jh}F_h$. This gives a well-defined map because $(*)$ continues to hold in T_B. Thus, we may take $R_{\lambda_0} = B$, and $S_0 = T_B/I_B$. We have that S_B is smooth over B, that $S = R \otimes_B S_B$. $\square$

(b) Let $P_1,\, \ldots\,, P_n$ be the minimal primes of R. Since R is reduced, their intersection is 0. Thus, R embeds in $\prod_{i=1}^{n}(R/P_i) \subseteq \prod_{i=1}^{n} \kappa_i$, where $\kappa_i = \mathrm{frac}(R/P_i)$. Since S is R-flat, $S = S \otimes_R R$ embeds in $S \otimes_R \prod_{i=1}^{n} \kappa_i \cong \prod_{i=1}^{n} \kappa_i \otimes_R S$, and the rings $\kappa_i \otimes_R S$ are the generic fibers and so are reduced. $\square$

(c) S is a localization of a smooth R-algebra, and so it suffices to show the result for smooth R-algebras, since a localization of a reduced ring is reduced. If $Q \in \mathrm{Spec}\,(S)$ is in the support of Sz, where z is nilpotent, then we may localize at the contraction P of S to R, and the image of z in S_P is a nonzero nilpotent. Now represent R_P as a direct limit as in part (a), and S as a direct limit of smooth extensions of these. Then each S_λ in the direct limit system is reduced, and it follows that S is reduced. $\square$

(d) The Henselization is a direct limit of pointed étale extensions. Each pointed étale extension is essentially smooth, and so reduced. Since a direct limit of reduced rings is reduced, the result follows. $\square$

2. The polynomial $Y^2 - X^2 - X^3$ is irreducible in $\mathbb{C}[X,Y]$, since $X^2 + X3 = X^2(1+X)$ is not a perfect square in the $\mathbb{C}[X]$. Hence, $\mathbb{C}[x,y] = \mathbb{C}[X,Y]/(Y^2 - X^2 - X^3)$ is a domain of dimension one, and $D = \mathbb{C}[x,y]_m$ where $m = (x,y)$ is a local domain of dimension one. In the Henselization, $1 + x$ has two square roots, f and $-f$, corresponding to the roots $1, -1$ of the equation $Z^2 - 1 = 0$. Then $(y - xf)(y + xf) = 0$ in the quotient, but neither factor is 0: there is a local $\mathbb{C}[[x]]$-homomorphism $D \to \mathbb{C}[[x]]$ that sends $y \mapsto \pm xg$ where $g = 1 + (1/2)x + \cdots$, where $g^2 = 1 + x$. One of these maps kills $y - xf$ but not $y + xf$, while the other kills $y + xf$ but not $y - xf$. $\square$

3. Let $Z = (Z_{ij})$ be an $s \times s$ matrix of unknowns. Then it suffices to show that the system of s^2 equations in s^2 unknowns given by the matrix equation $(I_s + Z)^n = I_s + M$, where M has entries in P, has a solution in P. Mod P, $Z = 0$ is a solution, and so it suffices to show that the $s^2 \times s^2$ Jacobian matrix has a determinant that is a unit. The equation may be written in matrix form as $nZ + Q(Z) - M = 0$ where $Q(Z)$ involves only terms that are of degree 2 and higher in the Z_{ij}. The Jacobian matrix is therefore nI_{s^2}, which will be invertible precisely when n is a unit.

4. Assume R is separable. Then for each minimal prime P of R, R_P is separable, and R_P is the fraction field of R/P. Hence, $R/P \subseteq \mathrm{frac}(R/P)$ is separable. In the other direction, since $R \hookrightarrow \prod_P (R/P)$ as P runs through the minimal primes of R, it suffices to show that $K \to \prod_P (R/P)$ is separable. Let K' be a finite purely inseparable extension of K. Then $K' \otimes_K (\prod_P R/P) \cong \prod_P (K' \otimes_K R/P)$ (this needs that K' is a finite-dimensional vector space over K) and since every factor is reduced, so is the product. Alternatively, one can use that for any field extension $L \supseteq K$, $L \otimes \prod_P (L \otimes_K R/P)$ is injective.
Note that, more generally, if V are vector spaces over K and W_j is a possibly infinite family of such vector spaces, we have a map $V \otimes_K \prod_j W_j \to \prod_j (V \otimes_K W_j)$ that sends $v \otimes \theta$ to the element whose j th coordinate is $v \otimes \theta_j$. If V is finite-dimensional with basis $v_1, \ldots, v_s$, this map is bijective: a typical element of $\prod_j (V \otimes_K W_j)$ has j th coordinate of the form $v_1 \otimes w_{1j} + \cdots v_s \otimes w_{sj}$, and the inverse takes this element to the sum of the of the elements $v_i \otimes \theta_i$, where θ_i has j th coordinate w_{ij}. If V is not finite-dimensional the map is injective, since for every finite-dimensional V_0 inside V, we have $V_0 \otimes_K \prod_j W_j \hookrightarrow \prod_j (V_0 \otimes_K W_j) \hookrightarrow \prod_j (V \otimes_K W_j)$ and the composite map is the same as V FINISH.

5. R must be Henselian. Let S be a module-finite extension of R. The maximal ideals of S/NS correspond bijectively to those of S, since NS consists of nilpotents, and since R/N is Henselian, S/NS decomposes as a product of quasilocal rings, one for each maximal ideal of S (or S/NS). Since idempotents lift uniquely modulo nilpotents, all of the idempotents yielding the product decomposition of S/NS lift to S, and so S also decomposes into such a product.

6. For the domain (resp., reduced) property, note that if $xy = 0$ (resp. $x^k = 0$) for nonzero x, y (resp., x) we have that x, y (resp., x) $\notin m^n \widehat{R}$ for large n. But then $XY = 0$ (resp. $X^k = 0$) has a solution in R that is congruent to the original solution modulo $m^n \widehat{R}$, which yields elements x_1, y_1 (resp., x_1) of R not in m^n such that $x_1 y_1 = 0$ (resp., $x_1^k = 0$). $\square$

EXTRA CREDIT 7. Since localizations are formally étale, every local ring of R is formally smooth over K and we may assume that R is local, since it suffices to check regularity locally. One has a coefficient field $L \subseteq R/P^2$ (which is complete), and by the structure theory for complete local rings, L contains K, since K is perfect. Let $x_1, \ldots, x_d$ be a minimal set of generators of P. Then we have a K-algebra surjection $R \twoheadrightarrow R/P^2 \cong L[X_1, \ldots, X_d]/m^2$, where $X_1, \ldots, X_d$ are indeterminates, $m = (X_1, \ldots, X_d)$, and the image of x_i maps to the image of X_i. Since R is formally smooth over K this lifts to a map $\theta_n : R \to L[X_1, \ldots, X_d]/m^{n+1} = S_n$ for all n. Let m_n be the maximal ideal of S_n. Since the images of the x_i generate m_n/m_n^2, by Nakayama's lemma they generate m_n. It follows from the part (c) of the Proposition stated at the bottom of p. 7 of the Supplement on the Structure Theory of Complete Local Rings that the map $R/P^n \to S_n$ is surjective. Hence, $\ell(R/P^n) \geq \ell(S_n)$, which is a polynomial of degree d in n for $n \gg 0$. It follows that $\dim(R) \geq d$, the least number of generators of P, while $\dim(R) \leq d$ is automatic. Thus, $\dim(R) = d$, and R is regular. $\square$

EXTRA CREDIT 8. We know from Problem **2.** that we may assume both rings are domains. It suffices to prove that if $\widehat{R}$ is not normal, then R is not normal. Suppose that y/x is an element of the fraction field of $\widehat{R}$ not in $\widehat{R}$, where $x \neq 0$. Then y/x satisfies a monic polynomial over $\widehat{R}$ of degree, say, n. Multiplying by x^n, we obtain a polynomial equation $(*)\quad y^n + z_1 y^{n-1}x + \cdots + z_i y^{n-i}x^i + \cdots + z_n x^n = 0$. Since $y/x \notin R$, we have that $y \notin xR$, and so there exists $N \gg 0$ such that $y \notin xR + m^N$ and $x \notin m^N$, where m is the maximal ideal of $\widehat{R}$. Let Y, X, Z_i be variables, and approximate the solution y, x, z_i of the equation $(**)\quad Y^n + Z_1 Y^{n-1}X + \cdots + Z_i Y^{n-i}X^i + \cdots + Z_n X^n = 0$ we have over $\widehat{R}$ by a solution in the approximation ring R, call it y', x', z_i', that is congruent to the original solution mod m^N. Then $y' \notin x'\widehat{R} + m^N$, and so $y \notin xR$,and $x' \notin m^N$, so that $x' \neq 0$. Hence, y'/x' is an element of the fraction field of R. Since y'/x' satisfies $(**)$, by dividing by x'^n we see that y'/x' is integral over R, a contradiction. $\square$