Math 615, Fall 2017 **Problem Set #5**
Due: Monday, April 17

1. Prove that if R and S are separable K-algebras, then $R \otimes_K S$ is separable.

2. Let $F = x^2y^2 + x^n + y^n \in A = \mathbb{C}[[x, y]]$ for $n \geq 5$, and let $B = A/(\partial F/\partial x, \partial F/\partial y)$. Show that B is an Artin local ring, finite-dimensional as a $\mathbb{C}$-vector space, such that the image of f of F in B is not 0, but that $xf = yf = 0$ in B, so that $f^2 = 0$ in B, and $df \in \Omega_{B/\mathbb{C}}$ is zero.

3. Let T be a formally smooth R-algebra, $I \subseteq T$ an ideal, and let $S = T/I$. Let $\overline{}$ indicate images mod I^2, and $[\,]$ mod I or mod $I\Omega_{T/R}$. The restriction of $d : T \to \Omega_{T/R}$ to I gives a map $I \to \Omega_{T/R}$ and, hence, $I \to \Omega_{T/R}/I\Omega_{T/R} \cong S \otimes_T \Omega_{T/R}$.

(a) Show that the map described kills I^2, and so induces a map $\eta : I/I^2 \to \Omega_{T/R}/I\Omega_{T/R}$ such that if $f \in I$, $\overline{f} \mapsto [df]$. Prove also that η is S-linear.

We know that for any surjection $T \to T/I = S$ of R-algebras, Ω_S is the quotient of $S \otimes_T \Omega_{T/R}$ by the R-span of the image of $\{df : f \in I\}$. Hence, there is an exact sequence $I/I^2 \xrightarrow{\eta} \Omega_{T/R}/I\Omega_{T/R} \to \Omega_{S/R} \to 0$ of S-modules.

(b) Show that giving a splitting, $\phi : T/I \to T/I^2$, as R-algebras, of the map $T/I^2 \twoheadrightarrow T/I$, is equivalent to giving, for all $t \in T$, an element $\delta(t) \in I/I^2$ such that $\phi([t]) = \overline{t} - \delta(t)$, subject to the conditions that $\delta : T \to I/I^2$ be an R-derivation and that $\delta(f) = \overline{f}$ for all $f \in I$. Thus, δ corresponds to a T-linear map $\theta : \Omega_{T/R} \to I/I^2$ such that $\delta = \theta \circ d$. Moreover, if $f \in I$ and $t \in T$, $\theta(f\,dt) = 0$, so that θ induces an S-linear map $\overline{\theta} : \Omega_{T/R}/I\Omega_{T/R} \to I/I^2$. With this notation, for all $f \in I$, $\overline{\theta}\big(\eta(\overline{f})\big) = \overline{f}$.

(c) Conclude that S is formally smooth over R iff $\eta : I/I^2 \to \Omega_{T/R}/I\Omega_{T/R}$ splits as a map of S-modules (hence, also, iff η is injective and $\Omega_{T/R}/I\Omega_{T/R} \twoheadrightarrow \Omega_{S/R}$ splits over S).

4. Let K be a field, and let $R = K[X_1, X_2, X_3, \ldots, X_n, \ldots]$ be the polynomial ring in countably many variables over K, let $m = (X_n : n \geq 1)R$, let S by the m-adic completion of R, and let $f_n = \sum_{j=n}^{\infty} x_j^j$. Is $f_n \in m^n S$? Prove your answer.

5. Let $\Lambda_0 = \mathbb{R}[t]_m \subseteq \mathbb{R}[[t]] = \Lambda$ where $m = t\mathbb{R}[t]$ and $\mathbb{R}$ is the real numbers. Let $S = \Lambda_0[t^a \cos t, t^b \sin t]$, where $a, b \geq 0$ are integers. Determine ℓ_S in terms of a and b, and describe explicit generators for the first Néron blow-up $S' = S[Q/t]$, where $Q = t\Lambda \cap S$.

6. Let D be a principal ideal domain such that $p_1D, \ldots, p_nD \ldots$ is an enumeration of the distinct maximal ideals of D (e.g., one may take $D = \mathbb{Z}$ or $\mathbb{Q}[x]$). Let G be the free D-module with free basis $e_1, \ldots, e_n, \ldots$, and let $f : G \to G$ be such that $f(e_n) = e_n - p_n e_{n+1}$ for all n. Let $C = \mathrm{Coker}\,(f)$, so that $(*)\ 0 \to G \xrightarrow{f} G \to C \to 0$ is exact. Let L be the fraction field of D. Let W be the set of square-fee elements of D. Show that $C \cong \{a/b : a \in D,\, b \in W\} \subseteq L$. Show that $(*)$ is locally split but not split.

EXTRA CREDIT 9. Let D be as in #6. and let $T = K[X_n : n \geq 1]$ be a polynomial ring in a countably infinite set of variables over D. Let $I = (X_n - p_n X_{n+1} : n \geq 1)T$, and $S = T/I$. Prove that S_Q is formally smooth over D for all $Q \in \mathrm{Spec}\,(S)$: in fact S_P is smooth over D_P for all $P \in \mathrm{Spec}\,(D)$. Prove that S is not formally smooth over D.

EXTRA CREDIT 10. (a) For $t \geq 1$, let B_t denote the tensor product over $\mathbb{C}$ of t copies of the ring constructed in Problem **2.** Show that B_t is an Artin local ring, and let f_j be the image of f from the j th copy of B in the tensor product , $1 \leq j \leq t$. Show that $g_t = \sum_{j=1}^{t} f_j \in B_t$ is an element of B_t such that $g_t^t \neq 0$ but $g_t^{t+1} = 0$.

(b) Let $A_t = \mathbb{C}[z]/z^t$. Show that $\mathrm{Hom}_{\mathbb{C}}(A_t, \mathbb{C}) \cong A_t$ as an A_t-module. Since $\mathrm{Hom}_{A_t}(M,\, A_t) \cong \mathrm{Hom}_{A_t}\bigl(M, \mathrm{Hom}_{\mathbb{C}}(A_t,\, \mathbb{C})\bigr) \cong \mathrm{Hom}_{\mathbb{C}}(M \otimes_{A_t} A_t,\, \mathbb{C})$ (adjointness of tensor and Hom) $\cong \mathrm{Hom}_{\mathbb{C}}(M, \mathbb{C})$ is an isomorphism of functors of M, conclude that A_t is injective in the category of A_t-modules and that every injection $A_t \hookrightarrow N$ of A_t-modules splits over A_t.

(c) Let R be any local Artin $\mathbb{C}$-algebra, and let u be an element such that $u^t \neq 0$ and $u^{t+1} = 0$, $t \geq 1$. Then $\mathbb{C}[u] \cong A_t$. Show that R injects into $S = R \otimes_{\mathbb{C}} B_t/(u - g_t)$, and that the image of u in S is such that $du = 0$ in $\Omega_{S/\mathbb{C}}$.

(d) Conclude that every Artin local ring over $\mathbb{C}$ that is a proper extension of $\mathbb{C}$ can be embedded in an increasing union of a sequence of Artin local $\mathbb{C}$-algebras in such a way that the differential of every element of the union is 0. The union is a non-Noetherian quasilocal $\mathbb{C}$-algebra in which every element of the maximal ideal is nilpotent, but if formally unramified over $\mathbb{C}$.